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A Recent Development Of The Equivariant Degree Methods And Their Applications In Symmetric Dynamical Systems

by

Haibo Ruan ©

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To my parents and my sister.

ABSTRACT

This work consists of three parts: theoretical preparations, applications and computational results. The theoretical part primarily addresses the basic properties and computational scheme of various equivariant degrees, including the general equivariant degree, primary equivariant degree, twisted primary degree, S^1-degree and equivariant gradient degree (Part I). The second part contains two types of applications of equivariant degree methods in the area of equivariant nonlinear analysis: the symmetric Hopf bifurcation and the existence of periodic solutions in autonomous symmetric systems (Part II). The last part presents an appendix of Sobolev spaces and a catalogue for several groups (their subgroups, irreducible representations and basic degrees), multiplication tables and computational results obtained throughout the thesis (Part III).

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1

Introduction

1.1 Motivation

The concept of symmetry, though as vague as the notion of harmony, and as simple as the impression evoked by regular geometric shapes, substantially contributes as a fundamental central theme to the ultimate design of nature. This has long been one idea that human beings have tried to comprehend in the pursuit of order, beauty and perfection (cf. [178, 166]). The idea of symmetry has been absorbed heuristically across human history, from architecture, to visual art, to psychology, to education science, to musicology and sociology, not to mention the interplay between arts and sciences (cf. [45, 151] and references therein).

In a sense, symmetry itself has been instrumental in the development of modern sciences. Some of the most profound results of modern physics have been underlined by symmetries. The duality between mass and energy, as well as between space and time, brought into light the special theory of relativity. As the drama of physics moved from the classical to the quantum act, symmetry was thrust into the limelight more than ever (cf. [186]). The mechanism of symmetry breaking embodies one of the most powerful ideas of modern theoretical physics. It provides a basis for most of the recent achievements in the description of phase transitions in statistical mechanics as well as of collective phenomena in solid state physics (cf. [148, 167, 183]). It has also made possible the understanding of the unification of weak, electromagnetic, and strong interactions in elementary particle physics (cf. [167, 184]).

Beyond the scope of modern physics, the presence of symmetry and its consequences have been extensively observed in chemistry, neurophysics, computer science, evolutionary ecology, sociology, and cognitive science (cf. [32, 56, 70, 82, 86, 161, 163, 171]). As the human perception naturally favors regular structure, elegant designs or artistic forms, symmetry has left traces in a large variety of dynamical systems (cf. [57, 76, 78] and references therein). While the symmetry may very well satisfy our expression of beauty and perfection, very little is known about the impact of symmetry on the performance of dynamical systems.

Nevertheless, there is inevitably a struggle in each symmetry with a tendency towards its breaking. The paradox resides in the failure of the symmetric laws of nature to establish a unified world. The observable phenomena exhibit overwhelmingly asymmetric diversity, which makes us believe that the natural processes are driven by a prize fight between the unified symmetry and diversified broken symmetries.

Examples of a win on the breaking side include the earthquake resistance failure of buildings, the occurrence of fluctuation in electrical circuits, crashes in electricity transmission networks, or environmental break-down in ecology models (cf. [13, 14, 15, 16, 77, 79, 81, 83]).

1.2 Area and Subject

Facing the enormous range of symmetry and the multitude of its impact, one seeks for a general understanding of the phenomena through a systematic and formal study. The mathematical treatment of symmetry is put forward in the language of group theory. Symmetry is understood as an intrinsic property of a mathematical object which causes it to remain invariant under certain groups of transformations, such as translation, rotation, reflection, inversion, or more abstract operations. A handful of Hermann Weyl's scientific works underscored group theory and its application in symmetry, including *The Theory of Groups and Quantum Mechanics*, *Classical Groups: Their Invariants and Representations*, and *Symmetry* (cf. [176, 177, 178]). As the struggle of symmetry persists, the process of mathematical abstraction of symmetry develops further, hoping to finally lead us to a mathematical understanding of great generality (cf. [174, 48, 91, 58]).

Behind the seemingly chaotic and overwhelmingly asymmetric natural phenomena, equivariant nonlinear analysis provides us with a kaleidoscope to the chromatic manifestation of symmetry. Equivariant dynamical systems are mathematical formalizations for a set of relations describing time-dependent processes of natural phenomena, which exhibit certain symmetric properties (cf. [34, 38, 78, 81, 159]). The equivariant nonlinear analysis serves for the study of equivariant dynamical systems and deals with the impact of symmetry on the existence, multiplicity, stability and topological structure of the solution set of nonlinear equations, bifurcation phenomena, the applicability of different kinds of approximation schemes, etc. (cf. [7, 15, 57, 77, 78]).

Traditional methods used in equivariant nonlinear analysis include center manifold and normal form reductions, Lyapunov-Schmidt reduction, algebraic and geometric formalism of Lie group theory and transformation group methods (cf. [33, 34, 80, 121, 122, 123]), minimax methods for nonconvex functionals, equivariant bifurcation theory, equivariant singularity theory, and theory of critical orbits of invariant functionals (cf. [57, 76, 77, 79, 81, 140, 159]).

The topological degree theory, without considerations of symmetry, has a long history which developed along successive steps of extensions and generalizations. The oldest form of a degree is probably the degree of a smooth map f from the unit circle S^1 into itself, also known as the "winding number", or the "rotational number", which counts the total number of times f travels counterclockwise around the origin. The mapping degree theory or its equivalent, the theory of rotation of vector fields, emerged in the studies of L. Kronecker and H. Poincaré, and was further developed in the works of L. Brouwer and H. Hopf for the finite-dimensional case, J. Leray and J. Schauder for completely continuous vector fields in infinite dimensional space (cf. [28, 29, 75, 128, 125]). It was however, M.A. Krasnosel'skii who indicated that knowing the mapping degree provides the answers to the qualitative theory of nonlinear operator equations (cf. [110]).

The degree theory of Brouwer and its infinite-dimensional extension — the Leray-Schauder degree, showed their weaknesses in certain circumstances related to the presence of symmetry. In particular, the Leray-Schauder degree often fails to detect periodic solutions in autonomous systems, due to the S^1-symmetry of periodic functions. A natural question arises: *what is an adequate theory of degree in the presence of symmetry?*

In 1932, K. Borsuk observed for the first time that symmetries can lead to restrictions on possible values of the degree, which then initiated a rigorous study of the impact of symmetry on the homotopical properties of the maps (cf. [24]). The subsequent developments were mainly due to P.A. Smith and M.A. Krasnosel'skii (cf. [23, 65]). At the same time, Krasnosel'skii revealed profound connections between the degree of equivariant maps and the equivariant extension problem (cf. [109]), which leads to a development of the so-called "geometric approach" (cf. [120] for the most recent results, and [101] where the case of linear abelian group actions is studied in detail) applying the concept of fundamental domains to reduce the problem of equivariant extensions to the case of free G-actions (cf. [120]) or fundamental cells (cf. [101]).

In 1967, Fuller defined a special index being a rational number known as the Fuller index, which was the first attempt to assign to an autonomous dynamical system an S^1-equivariant homotopy invariant (cf. [67]). Though of its theoretical importance, it is defined in an extended phase space, which makes this invariant difficult to compute. In 1988, G. Dylawerski introduced a degree theory for S^1-equivariant maps between representation spheres (cf. [51]). For a more general group of symmetries described by a compact Lie group G, a degree theory of G-equivariant maps was introduced by J. Ize *et al.* in [97] and rigorously studied in [101] for abelian groups. Independently, K. Gęba *et al.* constructed the S^1-degree using the idea of normal approximations, where connections between S^1-degree and the Fuller index were also indicated (cf. [52], see also [101]). Later, by applying similar constructions, a predecessor of the so-called primary equivariant degree for a compact Lie group G was introduced in [72]. Based on a result due to G. Peschke in [147], this primary degree can be recognized as the "primary part" of the equivariant degree introduced by Ize *et al.*

Contrary to the nonequivariant case, the homotopy structure of equivariant gradient maps is essentially different from those of non-gradient maps (cf. [146] for nonequivariant case and [41] for equivariant case). In 1985, motivated by the study of bifurcations of periodic solutions in Hamiltonian systems, E. N. Dancer introduced an invariant for S^1-equivariant gradient maps (cf. [40]). His idea of associating topological invariants to S^1-equivariant gradient fields, was further developed in several directions. In [44], E. N. Dancer and J. F. Toland introduced a topological invariant for systems with first integrals. S. Rybicki defined the S^1-degree for equivariant orthogonal maps as an extension of gradient maps in [153], which was generalized by J. Ize and A. Vignoli in [101] for abelian compact Lie groups. The gradient equivariant degree in the case of a general compact Lie group G, was introduced by K. Gęba in [71]. This degree takes values in the Euler ring $U(G)$, which is a generalization of the Burnside ring by T. tom Dieck (cf. [47]). This equivariant gradient degree contains implicitly the Dancer invariant which was mentioned above.

In comparison with traditional methods in equivariant nonlinear analysis, such as the equivariant Conley index, Morse-Floer complex, minimax theory (cf. [18, 19, 64, 134, 150, 172, 173]) and the equivariant singularity theory (cf. [57, 76, 77, 79, 81], see also [7, 15] and references therein), the equivariant degree theory has the following advantages: (cf. [5, 6, 10, 12, 13, 14, 17, 53, 55, 181, 118])

(a) Usage of the standard settings allowing efficient treatment of a large class of differential equations with arbitrarily large symmetry groups;
(b) Transparent computational formulae translating the equivariant spectral information of a linearized system into a topological invariant;
(c) Effective computerization* of algebraic computations, and creation of a database for classical symmetry groups collecting their subgroups, irreducible representations and multiplication tables;
(d) Comprehensive form of the topological invariant, which contains full topological information about the solution set of considered systems.

1.3 Two Examples

To demonstrate the mechanism of equivariant degree methods included in the thesis, we provide two examples of its application in equivariant nonlinear analysis. One looks into the Hopf bifurcation in a symmetric system of predator-prey equations; the other investigates the existence of periodic solutions in a symmetric Newtonian system.

1.3.1 Predation and Migration

Consider an ecosystem composed of 6 spatially symmetrically distributed subcommunities represented in Figure 1.1. Each subcommunity involves a predator-prey interaction between 2 species modeled by the Lotka-Volterra equations (with a slight modification), while the ecosystem is organized by a migration between every 2 adjacent subcommunities.

Recall the Lotka-Volterra equations, proposed independently by Alfred J. Lotka in 1925 and Vito Volterra [illegible] 1926 describe predation dynamics:

$$\begin{cases} \dot{x} = x(\alpha - \beta y), \\ \dot{y} = -y(\gamma - \delta x), \end{cases} \tag{1.1}$$

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* Special Maple © routines have been developed to assist effective computations of primary degree with one free parameter, for several interesting symmetry groups. The most recent version is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/~wkrawcew/degree`.

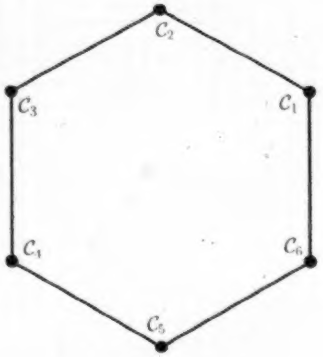

Fig. 1.1. Dihedral configuration of the ecosystem, where $C_i = (x_i(t), y_i(t))$ is the i-th community.

where $x = x(t)$ stands for the prey density and $y = y(t)$ for the predator density. The quantities α and γ corresponds to the intrinsic growth rate of the prey and the diminishing rate of the predator respectively; β and δ reflect the predation impact factors on the growth rate of the prey and the predator respectively. All the quantities $\alpha, \gamma, \beta, \delta$ are assumed to be positive.

Assume that the predator-prey interaction in each subcommunity can be modeled by a modified version of (1.1) (cf. [158, 66])

$$\begin{cases} \dot{x} = x(\alpha + cx - \beta y), \\ \dot{y} = -y(\gamma - \delta x - dy), \end{cases} \tag{1.2}$$

where c and d are parameters of returns. The case where c and d are both positive corresponds to the case of *increasing returns*, whereas the case where both are negative corresponds to *diminishing returns*. The case where c and d have opposite signs corresponds to *semi-increasing returns*. Biologically, increasing (resp. diminishing) returns in either species means that the growth of that species is enhanced (resp. hindered) by increasing the species density.

To carry out a mathematical analysis to the system (1.2), we first introduce the following shorthand notations

$$A = \alpha\delta + \gamma c, \quad B = \beta\gamma - \alpha d, \quad C := \delta\beta + dc.$$

For simplicity, assume that A, B and $C > 0$. Then, the system (1.2) has the following equilibria

$$(0,0),\ \left(-\frac{\alpha}{c},0\right),\ \left(0,\frac{\gamma}{d}\right),\ \left(\frac{B}{C},\frac{A}{C}\right),$$

among which the last one is called the *interior equilibrium*. We are interested in the phenomenon of the Hopf bifurcation taking place in the neighborhood of the interior equilibrium

$$(x_o(\alpha), y_o(\alpha)) = \left(\frac{B}{C}, \frac{A}{C}\right),$$

where α is the parameter of bifurcation. To obtain the characteristic roots of (1.2), we carry out the standard linearization of (1.2) at $(x_o(\alpha), y_o(\alpha))$ given by

$$\begin{cases} \dot{x} = \frac{cB}{C}x - \frac{\beta B}{C}y, \\ \dot{y} = \frac{\delta A}{C}x + \frac{dA}{C}y. \end{cases} \tag{1.3}$$

Denote by

$$M_o := \begin{bmatrix} \frac{cB}{C} & -\frac{\beta B}{C} \\ \frac{\delta A}{C} & \frac{dA}{C} \end{bmatrix}. \tag{1.4}$$

Then, the characteristic roots of (1.2) are precisely the eigenvalues of M_o. By the implicit function theorem, a necessary condition for an occurrence of a Hopf bifurcation at $(x_o(\alpha), y_o(\alpha))$ is that (1.2) has a purely imaginary characteristic root. The corresponding value α is called a *bifurcation center*. It can be verified that under the assumption $(dA + cB)^2 < 4ABC$, M_o has a pair of complex eigenvalues $\lambda(\alpha) = u(\alpha) \pm iv(\alpha)$ for

$$u(\alpha) = \frac{dA + cB}{2C}, \quad v(\alpha) = \frac{\sqrt{4ABC - (dA + cB)^2}}{2C}. \tag{1.5}$$

Solving $u(\alpha) = 0$ for α, we obtain the bifurcation center of (1.2) at $(x_o(\alpha), y_o(\alpha))$

$$\alpha_o := \frac{\gamma c(\beta + d)}{d(c - \delta)}.$$

We are now in a position to analyze the ecosystem composed of 6 subcommunities located in spatially symmetrically distributed habitats (cf. Figure 1.1). Assume that each subcommunity $\mathcal{C}_i$ undergoes a predator-prey interaction described by the modified Lotka-Volterra equations (1.2) with (x, y) replaced by (x_i, y_i), for $i = 1, 2, \ldots, 6$. The ecosystem supports a mild migration between every 2 adjacent subcommunities, with the migration rate given by $\nu > 0$. Hardly any communities in ecology are identical, residing in precise symmetrically distributed locations, however, when dealing with a model with accuracy-limited data, this idealization allows us to explore the symmetry aspect of the dynamics, including its impact on the occurrence of Hopf bifurcations in the system.

The mathematical description of this symmetric configuration is a symmetry with respect to the *dihedral group* D_6[*]. For a population density vector $w = (x_1, y_1, x_2, y_2, \dots, x_6, y_6)^T \in \mathbb{R}^{12}$, each element of D_6 acts as a linear transformation of w. More precisely,

$$\begin{cases} \mu(x_1, y_1, x_2, y_2, \dots, x_6, y_6) = (x_6, y_6, x_1, y_1, \dots, x_5, y_5), \\ \kappa(x_1, y_1, x_2, y_2, \dots, x_6, y_6) = (x_6, y_6, x_5, y_5, \dots, x_1, y_1). \end{cases}$$

Therefore, we consider an ecosystem of D_6-symmetrically located subcommunities of predator-prey interactions, which is modeled by

$$\begin{cases} \dot{x}_i = x_i(\alpha + cx_i - \beta y_i) + \nu(x_{i+1} - x_i) + \nu(x_{i-1} - x_i), \\ \dot{y}_i = -y_i(\gamma - \delta x_i - dy_i) + \nu(y_{i+1} - y_i) + \nu(y_{i-1} - y_i), \end{cases} \quad i = 1, 2, \dots, 6, \tag{1.6}$$

where x_i, y_i are the respective population density of the prey and predator in the i-th subcommunity, and $i \equiv i + 6$ for $i = 1, 2, \dots, 6$. It can be verified that the system (1.6) is invariant under the action of the dihedral group D_6, which is called a D_6*-symmetric* system. Our aim is to present a symmetric analysis of the Hopf bifurcation phenomena occurring in the system (1.6) at its interior equilibrium $w_o(\alpha)$, where

$$w_o(\alpha) := (x_o(\alpha), y_o(\alpha), x_o(\alpha), y_o(\alpha), \dots, x_o(\alpha), y_o(\alpha))^T \in \mathbb{R}^{12},$$

and α is the parameter of bifurcation.

The linearization of the system (1.6) at $w_o(\alpha)$ can be written as

$$\dot{w} = Mw + \nu \mathcal{C} w, \tag{1.7}$$

where $w = (x_1, y_1, x_2, y_2, \dots, x_6, y_6)^T$ is the population density vector, the matrix M represents the initial predation in subcommunities and the matrix $\mathcal{C}$ stands for the interaction relation between adjacent subcommunities

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[*] The group D_6 is composed of 6 rotations 1, μ, μ^2, μ^3, μ^4, μ^5, for $\mu = e^{i\frac{\pi}{3}}$ in the complex plane, and 6 reflections κ, $\kappa\mu$, $\kappa\mu^2$, $\kappa\mu^3$, $\kappa\mu^4$, $\kappa\mu^5$, where κ denotes complex conjugation (cf. Appendix A2.1.2 for more details).

$$M := \begin{bmatrix} M_o & 0 & 0 & 0 & 0 & 0 \\ 0 & M_o & 0 & 0 & 0 & 0 \\ 0 & 0 & M_o & 0 & 0 & 0 \\ 0 & 0 & 0 & M_o & 0 & 0 \\ 0 & 0 & 0 & 0 & M_o & 0 \\ 0 & 0 & 0 & 0 & 0 & M_o \end{bmatrix}, \quad \mathcal{C} := \begin{bmatrix} -2I & I & 0 & 0 & 0 & I \\ I & -2I & I & 0 & 0 & 0 \\ 0 & I & -2I & I & 0 & 0 \\ 0 & 0 & I & -2I & I & 0 \\ 0 & 0 & 0 & I & -2I & I \\ I & 0 & 0 & 0 & I & -2I \end{bmatrix},$$

for M_o defined by (1.4) and I being the 2×2 identity matrix. Let $\sigma(\mathcal{C})$ be the spectrum of $\mathcal{C}$, which contains 4 elements $\xi_0 := 0$, $\xi_1 := -1$, $\xi_2 := -3$ and $\xi_4 := -4$. Denote by $E(\xi_k)$ the eigenspace of ξ_k for $k = 0, 1, 2, 4$. Since $\mathcal{C}$ is a symmetric matrix, there exists an orthogonal linear transformation matrix P such that

$$\mathcal{C} = P \begin{bmatrix} \xi_0 I_{2\times 2} & 0 & 0 & 0 \\ 0 & \xi_1 I_{4\times 4} & 0 & 0 \\ 0 & 0 & \xi_2 I_{4\times 4} & 0 \\ 0 & 0 & 0 & \xi_4 I_{2\times 2} \end{bmatrix} P^T.$$

Moreover, the eigenspaces of $\mathcal{C}$ span the whole phase space

$$\mathbb{R}^{12} = E(\xi_0) \oplus E(\xi_1) \oplus E(\xi_2) \oplus E(\xi_4).$$

Since each $E(\xi_k)$ is invariant under the D_6 action, it is isomorphic to a sum of several copies of irreducible representations of D_6 (cf. Appendix A2.2.4 for a list). One can verify, in our case, we have $E(\xi_k) \simeq \mathcal{V}_k \oplus \mathcal{V}_k$, where $\mathcal{V}_k$ stands for the kth irreducible representation of D_6, for $k = 0, 1, 2, 4$. In particular, the $\mathcal{V}_l$-multiplicity of ξ_k is

$$m_l(\xi_k) = 2\delta_{l,k} = \begin{cases} 2, & \text{if } l = k \\ 0, & \text{otherwise} \end{cases}, \quad l, k \in \{0, 1, 2, 4\}.$$

By a change of coordinates $q := P^T w$, the system (1.7) is transformed to

$$\dot{q} = \widetilde{M} q, \tag{1.8}$$

where

$$\widetilde{M} := \begin{bmatrix} M_o + \nu\xi_0 I & 0 & 0 & 0 & 0 & 0 \\ 0 & M_o + \nu\xi_1 I & 0 & 0 & 0 & 0 \\ 0 & 0 & M_o + \nu\xi_1 I & 0 & 0 & 0 \\ 0 & 0 & 0 & M_o + \nu\xi_2 I & 0 & 0 \\ 0 & 0 & 0 & 0 & M_o + \nu\xi_2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & M_o + \nu\xi_4 I \end{bmatrix}. \tag{1.9}$$

Then, the characteristic roots of the system (1.6) at $w_o(\alpha)$ correspond to the eigenvalues of $\widetilde{M}$, which can be determined by

$$\begin{aligned}\mu_k(\alpha) &:= \lambda(\alpha) + \nu\xi_k, \\ &= u(\alpha) + \nu\xi_k \pm iv(\alpha) \quad k = 0, 1, 2, 4, \end{aligned} \tag{1.10}$$

where $u(\alpha)$ and $v(\alpha)$ are defined by (1.5). Clearly, $m_l(\mu_k(\alpha)) = m_l(\xi_k) = \delta_{l,k}$.

For each $\xi_k \in \sigma(\mathcal{C})$, by letting $u(\alpha) + \nu\xi_k = 0$, we obtain a bifurcation center

$$\alpha_k := \frac{\gamma c(\beta + d) + 2\nu\xi_k C}{d(c - \delta)}, \quad k = 0, 1, 2, 4.$$

Denote by $i\beta_k$ the purely imaginary characteristic root of (1.6) at $\alpha = \alpha_k$. To detect the occurrence of possible Hopf bifurcations around α_k, we associate to each pair (α_k, β_k) a *bifurcation invariant* $\omega(\alpha_k, \beta_k)$ in terms of a *twisted equivariant degree* for a *completely continuous field*.

More precisely[*], by introducing an additional parameter of the unknown period of possible bifurcating branches, we transform the system (1.6) to an equivalent problem of finding 2π-periodic solutions to a *normalized* system. Based on this normalization, we can choose an appropriate functional space W, which is an invariant space under the $D_6 \times S^1$-action, where $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ represents the temporal symmetries of 2π-periodic functions in W. Thereby, we reformulate the normalized system to a $D_6 \times S^1$-equivariant fixed-point problem of a *completely continuous map* $\mathcal{F} : \mathbb{R}^2 \oplus W \to W$, i.e. the problem of finding x such that $x = \mathcal{F}(\alpha, \beta, x)$. Finally, by introducing an *auxiliary function* $\varsigma : \mathbb{R}^2 \oplus W \to \mathbb{R}$, we are able to restrain the bifurcating branches in a neighborhood Ω of (α, β) so to carry out a local analysis of a one-parameter map $\mathfrak{F} : \mathbb{R}^2 \oplus W \to \mathbb{R} \oplus W$ composed by ς and $\mathcal{F}$. Define

$$\omega(\alpha_k, \beta_k) := D_6{\times}S^1\text{-Deg}(\mathfrak{F}, \Omega).$$

The computation of the bifurcation invariants is based on

▶ a continuous deformation of $\mathfrak{F}$ to a product map of $\overline{\mathfrak{F}} : \mathbb{R}^{12} \to \mathbb{R}^{12}$ and $\mathfrak{F}_o : \mathbb{R}^2 \oplus W_o \to \mathbb{R} \oplus W_o$, where $W_o := W \ominus \mathbb{R}^{12}$;

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[*] For a concise presentation, here we only provide a brief description of this standard degree-theoretical treatment to a symmetric Hopf bifurcation problem. For more technical details and precise formulations, we refer to Chapter 6.

- the multiplicativity property of the twisted equivariant degree, which implies (cf. Proposition 4.2.6)

$$\omega(\alpha_k,\beta_k) = D_6\times S^1\text{-Deg}(\overline{\mathfrak{F}},\overline{\Omega}) \circ D_6\times S^1\text{-Deg}(\mathfrak{F}_o,\Omega_o), \tag{1.11}$$

  where $\overline{\Omega} := \Omega\cap\mathbb{R}^{12}$, $\Omega_o := \Omega\cap(\mathbb{R}^2\oplus W_o)$ and $\circ$ stands for the $A(D_6)$-module multiplication in $A^t(D_6\times S^1)$ (cf. Appendix A3.14);
- the concept of *basic degrees* $\deg_{\mathcal{V}_i}$ (of no parameters) associated with the i-th irreducible representation $\mathcal{V}_i$ of D_6, combined with the negative spectrum σ_- of $\overline{\mathfrak{F}}$, gives rise to (cf. Subsection 4.1.3)

$$D_6\times S^1\text{-Deg}(\overline{\mathfrak{F}},\overline{\Omega}) = \prod_{\mu\in\sigma_-}\prod_i (\deg_{\mathcal{V}_i})^{m_i(\mu)}, \tag{1.12}$$

  where $m_i(\mu)$ is the $\mathcal{V}_i$-multiplicity of μ;
- the concept of *twisted basic degrees* $\deg_{\mathcal{V}_{j,l}}$ associated with the irreducible representation $\mathcal{V}_{j,l}$ of $D_6\times S^1$, combined with the notion of the *crossing numbers* $t_{j,l}(\alpha_k,\beta_k)$, gives rise to (cf. Subsection 4.2.4 and Lemma 3.3.4)

$$D_6\times S^1\text{-Deg}(\mathfrak{F}_o,\Omega_o) = \sum_{j,l} t_{j,l}(\alpha_k,\beta_k)\deg_{\mathcal{V}_{j,l}}. \tag{1.13}$$

Based on the computational formulae (1.11)–(1.13), we provide a computational example of the Hopf bifurcation problem for the system (1.6). Take the sample quantities

$$\gamma = 0.50,\ \beta = 1.00,\ \delta = 0.50,\ c = 0.20,\ d = -0.30, \nu = 0.01.$$

In this case, we have the following bifurcation centers

$$\alpha_0 = 0.78,\ \alpha_1 = 0.68,\ \alpha_2 = 0.48,\ \alpha_4 = 0.38,$$

with the corresponding purely imaginary roots $i\beta_k$

$$\beta_0 = 79.44,\ \beta_1 = 73.83,\ \beta_2 = 62.29,\ \beta_4 = 56.28.$$

The crossing numbers $t_{j,l}(\alpha_k,\beta_k)$ are

$$t_{0,1}(\alpha_0,\beta_0) = t_{1,1}(\alpha_1,\beta_1) = t_{2,1}(\alpha_2,\beta_2) = t_{4,1}(\alpha_4,\beta_4) = 2.$$

Consequently, we have the values of bifurcation invariants $\omega(\alpha_k,\beta_k) = 2\deg_{\mathcal{V}_{k,1}}$ for $k = 0,1,2,4$. Calling the Maple© routine command `showdegree[D6]`, we obtain

$$\begin{aligned}
\omega(\alpha_0, \beta_0) &= \texttt{showdegree[D6]}(0,0,0,0,0,0,2,0,0,0,0,0)\\
&= 2(\mathbf{D}_6),\\
\omega(\alpha_1, \beta_1) &= \texttt{showdegree[D6]}(0,0,0,0,0,0,0,2,0,0,0,0)\\
&= 2(\mathbb{Z}_{\mathbf{6}}^{\mathbf{t_1}}) + 2(D_2^d) + 2(\mathbf{D}_2^{\hat{d}}) - 2(\mathbb{Z}_2^-)\\
\omega(\alpha_2, \beta_2) &= \texttt{showdegree[D6]}(0,0,0,0,0,0,0,0,2,0,0,0)\\
&= 2(\mathbb{Z}_{\mathbf{6}}^{\mathbf{t_2}}) + 2(\mathbf{D}_2^z) + 2(D_2) - 2(\mathbb{Z}_2)\\
\omega(\alpha_4, \beta_4) &= \texttt{showdegree[D6]}(0,0,0,0,0,0,0,0,0,0,2,0)\\
&= 2(\mathbf{D}_6^d),
\end{aligned}$$

where each $(H^{\delta,m})$ refers to the conjugacy class of the subgroup $H^{\delta,m} \subset D_6 \times S^1$ (cf. Example A2.1.1, Appendix A2) and the bold faced terms are related to the concept of *dominating orbit types*, which satisfy certain maximality condition according to the conjugation relation (cf. Subsection 6.1.6).

Conclusion. By Theorem 6.1.9 (ii), there exists at least **1** bifurcating branch of periodic solutions at (α_o, β_o) of symmetry at least $(\mathbf{D_6})$; there exist at least **5** bifurcating branches of periodic solutions at (α_1, β_1) with **2** having symmetry at least $(\mathbb{Z}_{\mathbf{6}}^{\mathbf{t_1}})$ and **3** having symmetry at least $(\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}}})$; there exist at least **5** bifurcating branches of periodic solutions at (α_2, β_2) with **2** having symmetry at least $(\mathbb{Z}_{\mathbf{6}}^{\mathbf{t_2}})$ and **3** having symmetry at least $(\mathbf{D}_{\mathbf{2}}^{\mathbf{z}})$; there exists at least **1** bifurcating branch of periodic solutions at (α_4, β_4) with symmetry at least $(\mathbf{D}_{\mathbf{6}}^{\mathbf{d}})$.

Evidently, when α crosses the bifurcation centers α_k for $k = 0, 1, 2, 4$, the total symmetry $D_6 \times S^1$ of the trivial solution $w \equiv 0$ breaks down to different subsymmetries (D_6), $(\mathbb{Z}_6^{t_1})$, $(D_2^{\hat{d}})$, $(\mathbb{Z}_6^{t_2})$, (D_2^z), (D_6^d) of nonzero solutions. The broken symmetries captured by the invariants $\omega(\alpha_k, \beta_k)$, in turn, entail the appearance of nontrivial periodic solutions bifurcating from $\alpha = \alpha_k$.

1.3.2 Newtonian Motions

Consider a system of 6 unit point masses P_i (for $i = 1, 2, \dots, 6$) trajecting in $\mathbb{R}^6$, whose time-dependent position function $x : \mathbb{R} \to \mathbb{R}^6$ satisfies Newton's second law of motion, which states that the time rate of change of the velocity function is proportional to a net force function $F : \mathbb{R}^6 \to \mathbb{R}^6$ applied on the particles. Suppose that the system is autonomous and *symmetric* with respect to the dihedral group D_6-action on $\mathbb{R}^6$. More precisely, F is assumed to commute with the D_6-action on $\mathbb{R}^6$, i.e. $F(gx) = gF(x)$ for $g \in D_6$ and $x \in \mathbb{R}^6$.

An example of such autonomous Newtonian system can be described by

$$\begin{cases} -\ddot{x}_1 = 4x_1 + x_2 + x_6 + a(x)(5x_1 + x_2 + x_6), \\ -\ddot{x}_2 = x_1 + 4x_2 + x_3 + a(x)(x_1 + 5x_2 + x_3), \\ -\ddot{x}_3 = x_2 + 4x_3 + x_4 + a(x)(x_2 + 5x_3 + x_4), \\ -\ddot{x}_4 = x_3 + 4x_4 + x_5 + a(x)(x_3 + 5x_4 + x_5), \\ -\ddot{x}_5 = x_4 + 4x_5 + x_6 + a(x)(x_4 + 5x_5 + x_6), \\ -\ddot{x}_6 = x_5 + 4x_6 + x_1 + a(x)(x_5 + 5x_6 + x_1), \end{cases} \tag{1.14}$$

where $a : \mathbb{R}^6 \to \mathbb{R}$ is given by $a(x) = \big(5(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) + 2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1) + 1\big)^{-\frac{3}{2}}$.

We are interested in finding non-constant 2π-periodic solutions to (1.14), which can be formulated precisely as finding nontrivial solutions to

$$\begin{cases} -\ddot{x} = F(x), \\ x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{1.15}$$

where the force function $F : \mathbb{R}^6 \to \mathbb{R}^6$ is represented by the right-hand side of (1.14). Notice that F behaves *asymptotically linear* at ∞, meaning that there exists a linear map A_∞ such that $F(x) = A_\infty x + o(x)$ as $\|x\| \to \infty$. Let $A_0 := DF(0)$. We have

$$A_0 = \begin{pmatrix} 9&2&0&0&0&2 \\ 2&9&2&0&0&0 \\ 0&2&9&2&0&0 \\ 0&0&2&9&2&0 \\ 0&0&0&2&9&2 \\ 2&0&0&0&2&9 \end{pmatrix}, \quad A_\infty = \begin{pmatrix} 4&1&0&0&0&1 \\ 1&4&1&0&0&0 \\ 0&1&4&1&0&0 \\ 0&0&1&4&1&0 \\ 0&0&0&1&4&1 \\ 1&0&0&0&1&4 \end{pmatrix},$$

which represent the linearized maps of F at 0 and ∞ respectively. Further, one verifies that $(\sigma(A_0) \cup \sigma(A_\infty)) \cap \{k^2 \;:\; k = 0, 1, \dots\} = \emptyset$, which eliminates the possibility of the linearized systems of (1.15) at 0 and ∞ having non-zero solutions. Therefore, it provides an admissible setting to detect a nontrivial solution to (1.15) by inspecting the topological difference between the linearized systems of (1.15) at 0 and at ∞.

More precisely*, choose an appropriate functional space W, where the solutions to (1.15) inhabit, and reformulate (1.15) as a $D_6 \times S^1$-equivariant *vari-*

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[*] We only provide a concise description of the degree-theoretical treatment to a symmetric variational problem. For more technical details and precise formulations, we refer to Chapter 10.

ational problem of finding critical points of a certain associated energy functional $\Phi : W \to \mathbb{R}$, where $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ represents the temporal symmetries of the 2π-periodicity of functions in W. Thus, we have

$$x \text{ is a solution to (1.15)} \quad \Longleftrightarrow \quad \nabla\Phi(x) = 0,\ x \in W.$$

By a compactness argument, $\nabla\Phi$ is a $D_6 \times S^1$-equivariant *completely continuous field*, to which the equivariant degree theory applies. By the spectral properties of A_0 and A_∞, combined with the implicit function theorem, there exists a small ball B_ε and a large ball B_R in W such that $\nabla\Phi(x) \neq 0$ for [illegible] boundary points $x \in (\partial B_\varepsilon \cup \partial B_R)$. By means of the *gradient equivariant degree* $\nabla_{D_6\times S^1}$-deg, we can associate to the system (1.15) two elements $\nabla_{D_6\times S^1}$-deg$(\nabla\Phi, B_\varepsilon)$ and ∇_G-deg$(\nabla\Phi, B_R)$ in a ring called the *Euler ring* $U(D_6 \times S^1)$. Roughly speaking, in the context of the ring, one is allowed to multiply two gradient equivariant degrees. Therefore, the difference

$$\deg_\infty - \deg_0 := \nabla_{D_6\times S^1}\text{-deg}(\nabla\Phi, B_R) - \nabla_{D_6\times S^1}\text{-deg}(\nabla\Phi, B_\varepsilon)$$

is a *topological invariant* capturing the existence of solutions to (1.15) in between B_ε and B_R.

Computational techniques used for $\deg_\infty - \deg_0$ are based on

- the linearization argument, which relays the computations of $\deg_p$ to the computations of a linear isomorphism $\mathcal{A}_p$ on W by

  $$\deg_p = \nabla_{D_6\times S^1}\text{-deg}(\mathcal{A}_p, B_1(W)),$$

  where $B_1(W)$ denotes the unit ball in W and $\mathcal{A}_p : W \to W$ is defined through A_p, for $p \in \{0, \infty\}$;
- the reduction to the *basic gradient degrees*, denoted by

  $$\mathrm{Deg}_{\mathcal{W}} := \nabla_{D_6\times S^1}\text{-deg}(-\mathrm{Id}, B_1(\mathcal{W})),$$

  for an irreducible representation $\mathcal{W}$ of $D_6 \times S^1$. Observe that $\mathrm{Deg}_{\mathcal{W}}$'s represent the gradient equivariant degrees of the simplest possible maps being topologically nontrivial (cf. Definition 5.2.7);
- the multiplicity property of the gradient equivariant degree (inherited from the ring multiplication in the Euler ring $U(D_6 \times S^1)$), induces a product formula (cf. Subsection 5.2.2)

$$\deg_p = \prod_{\xi\in\sigma_-(\mathcal{A}_p)} \prod_k (\mathrm{Deg}_{\mathcal{W}_k})^{m_k(\xi)}, \tag{1.16}$$

where $\sigma_-(\mathcal{A}_p)$ stands for the negative spectrum of $\mathcal{A}_p$, $\mathcal{W}_k$ runs through a catalogue of irreducible $D_6\times S_1$-representations and $m_k(\xi)$ is the multiplicity of ξ with respect to $\mathcal{W}_k$. Notice that the product in (1.16) is only essential over finitely many terms, since $\mathcal{A}_p$ is a compact perturbation of Id, there exist only finitely many $\xi \in \sigma_-(\mathcal{A}_p)$ each of which has a finite multiplicity.

By calling the Maple© routine command `showdegree[D6]`, we obtain

$$\begin{aligned}\deg_\infty - \deg_0 = {}& -(\mathbf{D}_{\mathbf{6}}^{\mathbf{d},\mathbf{3}}) - (\mathbb{Z}_{\mathbf{6}}^{\mathbf{t}_1,\mathbf{3}}) + (D_3^3) + 3(D_2^{d,3}) + (\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}},\mathbf{3}}) + (\mathbb{Z}_3^{t,3})\\ & - 2(\widetilde{D}_1^{z,3}) - (\widetilde{D}_1^3) - (D_1^3) - 2(\mathbb{Z}_2^{-,3}) + 2(\mathbb{Z}_1^3) - (\mathbf{D}_{\mathbf{6}}^{\mathbf{d},\mathbf{2}})\\ & - (\mathbb{Z}_{\mathbf{6}}^{\mathbf{t}_2,\mathbf{2}}) + (D_3^2) + (\mathbf{D}_{\mathbf{2}}^{\mathbf{z},\mathbf{2}}) + 2(D_2^{d,2}) + (D_2^2) + (\mathbb{Z}_3^{t,2})\\ & - 2(\widetilde{D}_1^{z,2}) - (\widetilde{D}_1^2) - (D_1^2) - (\mathbb{Z}_2^{-,2}) + 2(\mathbb{Z}_1^2),\end{aligned} \tag{1.17}$$

where each $(H^{\delta,m})$ refers to the conjugacy class of the subgroup $H^{\delta,m} \subset D_6\times S^1$ (cf. Example A2.1.1, Appendix A2) and the highlighted terms are related to the concept of *dominating orbit types*, which satisfy certain maximality condition according to the conjugation relation.

Conclusion.

Based on the value of the invariant provided by (1.17), we conclude that there exist **11** nonconstant periodic solutions to (1.15), include **1** nonconstant solution of symmetry at least $(\mathbf{D}_{\mathbf{6}}^{\mathbf{d},\mathbf{3}})$, **2** nonconstant solutions of symmetry at least $(\mathbb{Z}_{\mathbf{6}}^{\mathbf{t}_1,\mathbf{3}})$, **3** nonconstant solutions of symmetry at least $(\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}},\mathbf{3}})$, **2** nonconstant solutions of symmetry at least $(\mathbb{Z}_{\mathbf{6}}^{\mathbf{t}_2,\mathbf{2}})$, and **3** nonconstant solutions of symmetry at least $(\mathbf{D}_{\mathbf{2}}^{\mathbf{z},\mathbf{2}})$.

Eminently, the initial symmetry $D_6 \times S^1$ of the stationary solution breaks down to several subsymmetries of other physical states (nonconstant periodic solutions), namely $(D_6^{d,3})$, $(\mathbb{Z}_6^{t_1,3})$, $(D_2^{\hat{d},3})$, $(\mathbb{Z}_6^{t_2,2})$ and $(D_2^{z,2})$, being captured by $\deg_\infty - \deg_0$.

1.4 Overview and Contribution

There are several different kinds of equivariant degrees appearing in the thesis: the general equivariant degree, primary equivariant degree, S^1-equivariant degree, twisted primary degree, equivariant gradient degree and orthogonal degree. Belonging to the same family of equivariant degrees, they are interconnected to each other.

Let G be the compact Lie group of symmetries. The general equivariant degree, usually denoted by $\deg_G$, produces the primary equivariant degree as its truncated part, which is written as $G\text{-Deg}$. In turn, the twisted primary degree $G\text{-Deg}^t$, is included as a twisted part of the primary degree, in the case $G = \Gamma \times S^1$ with Γ being a compact Lie group. The S^1-equivariant degree is a special case of the twisted degree for $G = S^1$, and often written as $S^1\text{-Deg}$. On the other hand, the equivariant gradient degree denoted by $\nabla_G\text{-deg}$, is an equivariant degree specially designed for gradient maps. It should be pointed out that $\nabla_G\text{-deg}$ generally differs from $\deg_G$, which is due to the fact that contrary to the non-equivariant case, the homotopy classes of gradient equivariant maps do not coincide with those of general equivariant maps. However, in the case of G being a one-dimensional bi-orientable compact Lie group, there exists a passage from $\nabla_G\text{-deg}$ to $G\text{-Deg}$, through yet another equivariant degree, namely the orthogonal degree $G\text{-Deg}^o$.

The equivariant degree introduced in [97], though of great importance in theory, provides no generous hints of its computations in practice. In contrast, the *primary equivariant degree* (with n-free parameters) shows a more efficient aspect in its computational perspective.

In Chapter 3, we propose an axiomatic definition of the primary equivariant degree, which lays down a convenient pavement for the usage of the primary degree outside the context of its topological origins (cf. Proposition 3.2.5). In particular, the primary equivariant degree with one free parameter proves to be completely computable. Based on an axiomatic definition of the S^1-equivariant degree (cf. Theorem 3.4.4) and a recurrence formula (cf. Proposition 3.5.3), the computations of primary G-equivariant degree (with one-free parameter) can be systematically reduced to those of related S^1-equivariant degrees.

Motivated by the study of symmetric Hopf bifurcation problems and the existence of periodic solutions in symmetric autonomous systems, we explore further properties of the primary equivariant degree for $G = \Gamma \times S^1$, where the compact Lie group Γ describes the spatial symmetry in considered dynamical systems.

There are two types of subgroups in $\Gamma \times S^1$: the *non-twisted* subgroups $K \times S^1$, and the *twisted* subgroups $K^{\varphi,l}$, where $K \subset \Gamma$ (cf. Definition 4.2.1). As nontrivial periodic functions only admit twisted subgroups of symmetries, it is natural to introduce the *twisted primary degree* as the twisted part of the primary equivariant degree, so to capture the presence of nontrivial periodic solutions to the dynamical systems (cf. Chapter 4).

The twisted primary degree stands out as the most efficient topological tool above others, contributing to the computerization of the equivariant degree method. Several Maple© routines[*] have been developed to enhance the speed of the algebraic computations. The computability of the twisted equivariant degree highly depends on its *multiplicativity* property, which is related to certain module structure on its range (cf. Proposition 4.2.6). Examples of multiplication tables for several groups are included in Appendix A3. By the multiplicity property, the computations of the twisted primary degree can be significantly reduced to the evaluations of the *twisted basic degrees* (cf. Definition 4.2.8). In Appendix A2, we prepare a catalogue of selected groups, their irreducible representations and corresponding twisted basic degrees.

The equivariant gradient degree introduced by K. Gęba, is an equivariant degree theory specially designed for variational problems (cf. [71]). In Chapter 5, our discussion starts with the range of the equivariant gradient degrees, namely, the *Euler ring*. Though the gradient degree inherits a natural multiplicativity property from the ring structure, it is generally difficult to be determined due to the complexity of the Euler ring multiplicative structure. However, as proved in Subsection 5.1.1, there exists a close relation between the Euler ring and the module structures arising from the primary degree theory (cf. Remark 5.1.13). Therefore, we speculate a possibility to construct a passage from the gradient degree to the primary degree, in order to make the computational resources available for the computations of the gradient degree. It turns out that in the case where G is a one-dimensional bi-orientable compact Lie group (cf. [147]), such a passage is possible through a construction of the *equivariant orthogonal degree*. Consequently, we establish computational formulae of equivariant gradient degrees based on the passage (cf. Subsection 5.2.4).

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[*] Current routines are available for quaternionic group Q_8, dihedral groups D_N, for $N = 3, 4, 5, 6, 8, 10, 12$, the tetrahedral group A_4, octahedral group S_4 and icosahedral group A_5. The most recent version is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/~wkrawcew/degree`.

In Chapter 6 — Chapter 8, we apply the twisted primary degree method to the study of Hopf bifurcations in equivariant dynamical systems. Roughly speaking, a Hopf bifurcation is the phenomenon occuring around a stationary solution $x = x_o$ to the system, undergoing a sudden change of stability, as a parameter α crosses some critical value α_o and thereby resulting in appearance of small amplitude nontrivial periodic solutions near x_o. In the language of symmetry, the Hopf bifurcation precisely refers to the moment when the whole symmetry of the stationary solution breaks down to smaller subsymmetries of the nontrivial periodic solutions.

To study the bifurcation phenomena, we associate a *bifurcation invariant* to each bifurcation center, by means of the twisted primary degree method. The nontrivial value of the invariant provides a sufficient condition for an appearance of Hopf bifurcations, and offers a *symmetric classification* of the bifurcating branches indicating their least symmetries. Further, if the invariant contains a nonzero $(K^{\varphi,l})$-term for a *dominating orbit type* $(K^{\varphi,1})$ (cf. Definition 6.1.7), then the exact symmetries of the bifurcating periodic solutions can be detected (after rescaling the period). The main advantage of this method is that it can be applied to different classes of equations in a standard manner (cf. functional differential equations in Chapter 6, neutral functional differential equations in Chapter 7 and the functional parabolic differential equations in Chapter 8). The computational examples are listed in Appendix A4.1—A4.3 for selected groups of symmetries.

In Chapter 9 and Chapter 10, we study the existence of nontrivial periodic solutions in equivariant autonomous dynamical systems. More precisely, in Chapter 9, we consider a symmetric Lotka-Volterra type system with delays, which arises naturally from an ecological model of symmetrically located predator-prey interactions. As this symmetric system falls out of the category of symmetric variational problems, only few topological methods are traditionally used. Unfortunately, some of those methods such as Leray-Schauder degree, are ineffective for detecting nontrivial periodic solutions.

By introducing additional homotopy parameters to the system and establishing *a priori* bounds for the parameterized systems, we are able to define a topological invariant as a twisted primary degree (cf. Definition 9.1.1), which detects the existence of multiple nontrivial periodic solutions to the original system (cf. Theorem 9.1.2). Indeed, the appearance of different nontrivial periodic solutions is engraved in the value of the topological invariants by their broken subsymmetries.

It is appropriate to mention that the main content of the thesis is based on several published journal papers co-authored by the author, namely [6, 11, 12, 13, 14, 68, 88, 152], and the catalogue of the groups and their representations is excerpted from [15]. Consequently, the scientific results included in the thesis originate from collaborative research rather than being an individual achievement.

1.5 Future Research

The methods and applications of the equivariant degree theory are far from being complete. For a general equivariant degree, a development of the computational methods for secondary equivariant degrees is needed. In the case of primary degree and twisted primary degree, multiparameter cases should be further explored, as well as their further connection with other equivariant degrees. To expand the applications of the gradient equivariant degree, we must establish effective methods for computation of Euler ring and basic gradient degrees, including new data base for other interesting groups such as $SO(3) \times S^1$, $U(2)$, $U(2) \times S^1$.

One can explore further potential applications to the existence of periodic solutions in autonomous systems based on the *a priori* bounds techniques. Another interesting phenomenon is forced symmetry breaking, which takes place when the total symmetry G of the system reduces to a smaller symmetry G_o under an asymmetric perturbation (cf. [34, 101]). By studying the homomorphism $U(G) \to U(G_o)$ (resp. $A(G) \to A(G_o)$) induced by the inclusion map $G_o \hookrightarrow G$, it is possible to determine the equivariant degrees of the perturbed system, thus allowing us to predict the forced symmetries of the system. Also, it is interesting to study the global continuation of branches of solutions by means of equivariant degree method, so as to have a global picture of the behavior of orbits of periodic solutions. Not the least, we can also investigate the bifurcation from relative equilibria, doubly periodic and Hopf bifurcations from periodic orbits (cf. [78]).

Part I

EQUIVARIANT DEGREE METHODS: TECHNICAL TOOLS

2

Preliminaries

2.1 Basic Facts from Differential Topology

2.1.1 Smooth Manifolds

Throughout, a *smooth manifold* always means a separable paracompact C^∞-smooth finite-dimensional manifold, and a *smooth map* between two manifolds is assumed to be of class C^∞. For smooth manifolds M, N and a smooth map $f : M \to N$, we denote by $\tau(M)$ the *tangent bundle* of M and $\tau_x(M)$ the *tangent space* of M at $x \in M$; $df : \tau(M) \to \tau(N)$ stands for the *tangent map* of f with $df_x : \tau_x(M) \to \tau_{f(x)}(N)$.

Definition 2.1.1. Let $f : M \to N$ be a smooth map between smooth manifolds. A point $x \in M$ is said to be *regular* if the rank of the induced map of tangent spaces $df_x : \tau_x(M) \to \tau_{f(x)}(N)$ is equal to $\dim N$; otherwise x is called *critical*. A point $y \in N$ is called a *regular value* of f if $f^{-1}(y)$ does not contain a critical point; otherwise y is called a *critical value* of f. By definition, y is a regular value if $f^{-1}(y) = \emptyset$.

The concept of a regular value naturally extends to the notion of a map transversally regular on a submanifold. More precisely, we have:

Definition 2.1.2. Let P be a smooth submanifold of a smooth manifold N and $k = \dim N - \dim P$ be the *co-dimension* of P in N, denoted by $\operatorname{codim}_N P$. Then, a smooth map $f : M \to N$ is said to be *transversally regular* with respect to P, if for every $x \in f^{-1}(P)$, the rank of the map

$$\tau_x(M) \xrightarrow{df_x} \tau_{f(x)}(N) \longrightarrow \tau_{f(x)}(N)/\tau_{f(x)}(P)$$

is maximal, i.e. equals to k.

Let us recall several well-known results.

Proposition 2.1.3. *If $f : M \to N$ is transversally regular with respect to $P \subset N$, then the complete inverse image $f^{-1}(P)$ is a smooth submanifold of M and* $\operatorname{codim}_M f^{-1}(P) = \operatorname{codim}_N P$, *whenever $f^{-1}(P) \neq \emptyset$.*

Corollary 2.1.4. *Let $f : M \to N$ be a smooth map and $y \in N$ a regular value of f. Then, $f^{-1}(y)$ is a* $(\dim M - \dim N)$*-dimensional smooth submanifold of M, whenever $f^{-1}(y) \neq \emptyset$.*

Proposition 2.1.5. (SARD-BROWN THEOREM) *Let M be a smooth compact manifold and $f : M \to \mathbb{R}^k$ a smooth map. Then, the set of all critical values of f has Lebesgue measure zero in $\mathbb{R}^k$. Moreover, the set of all regular values of f is open and dense in $\mathbb{R}^k$.*

Corollary 2.1.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}^k$ a smooth map and $K \subset \Omega$ a compact subset. Take $y \in \mathbb{R}^k$ and $\varepsilon > 0$. There exists a smooth map $g : \Omega \to \mathbb{R}^k$ such that y is a regular value of g and*

$$\sup\{\|f(x) - g(x)\| : x \in K\} < \varepsilon.$$

Another important consequence of the Sard-Brown theorem is related to the realization of compact manifolds as submanifolds in $\mathbb{R}^N$. To be more specific,

Definition 2.1.7. A smooth map $f : M \to N$ is called *embedding* if the following two conditions are satisfied:

(i) the rank of the induced map $df_x : \tau_x(M) \to \tau_{f(x)}(N)$ is $\dim M$ for all $x \in M$ (in particular, we must then have $\dim M \leq \dim N$);
(ii) $f : M \to f(M)$ is a homeomorphism.

We have:

Proposition 2.1.8. (WHITNEY THEOREM) *Let M be a compact n-dimensional manifold (possibly with boundary). Then:*

(i) M can be embedded into $\mathbb{R}^{2n+1}$;
(ii) if $g : M \to \mathbb{R}^{2n+1}$ is a continuous map and $\varepsilon > 0$, then there exists an embedding $f : M \to \mathbb{R}^{2n+1}$ such that

$$\sup\{\|f(x) - g(x)\| : x \in M\} < \varepsilon.$$

2.1.2 Oriented Vector Bundles

For an n-dimensional vector space V, we say that two ordered bases $\mathfrak{b}_1 := (b_1, b_2, \dots, b_n)$ and $\mathfrak{b}_2 := (b'_1, b'_2, \dots, b'_n)$ of V determine the *same orientation* of V if the change-of-coordinates matrix from $\mathfrak{b}_1$ to $\mathfrak{b}_2$ has positive determinant. An *orientation* in V, denoted by $\mathfrak{o}_V$, is a class of all ordered bases $\mathfrak{b}$, which determine the same orientation in V. The pair $(V, \mathfrak{o}_V)$ is called an *oriented vector space* with the orientation $\mathfrak{o}_V$ on V. There are only two possible orientations of V, the other orientation of V is denoted by $-\mathfrak{o}_V$. The chosen orientation $\mathfrak{o}_V$ will be called *positive* and a basis $\mathfrak{b}$ representing $\mathfrak{o}_V$ is called *positive basis* in V. For a zero-dimensional vector space we adopt the convention to assign $+1$ (resp. -1) to indicate the positive (resp. negative) orientation. The orientation $\mathfrak{o}_n$ of the space $\mathbb{R}^n$, determined by the standard basis $(e_1, \dots, e_n)$ in $\mathbb{R}^n$, is called the *standard orientation* of $\mathbb{R}^n$.

For two oriented vector spaces $(V, \mathfrak{o}_V)$ and $(W, \mathfrak{o}_W)$, we denote by $\mathfrak{o}_V \odot \mathfrak{o}_W$ the *natural orientation* of the space $V \oplus W$ (i.e. the orientation represented by a positive basis of V followed by a positive basis of W) and we write $\mathfrak{o}_{V \oplus W} := \mathfrak{o}_V \odot \mathfrak{o}_W$. For an oriented vector space $(V, \mathfrak{o}_V)$, the vector space $\mathbb{R}^n \oplus V$ is always assumed to have the orientation $\mathfrak{o}_n \odot \mathfrak{o}_V$. For two oriented vector spaces $(V, \mathfrak{o}_V)$ and $(W, \mathfrak{o}_W)$ of the same dimension, a linear isomorphism $A : V \to W$ is said to *preserves the orientations of* V *and* W if a matrix representation of A, with respect to positive bases in V and W has positive determinant. In what follows, instead of writing $(V, \mathfrak{o}_V)$ we will simply say that V is an *oriented vector space*, what will implicitly mean that there is a chosen orientation $\mathfrak{o}_V$ on the space V.

Let $\xi = (p, E, B)$ be a vector bundle modeled on $\mathbb{R}^n$. Suppose that for every $x \in B$, it is possible to choose an orientation class $\mathfrak{o}_x$ in the fiber $p^{-1}(x)$ in such a way that there exists a family $\{(U_i, \varphi_{U_i})\}$ of local trivializations of ξ satisfying $B = \bigcup_i U_i$ and such that:

(i) for all $x \in U_i$, the linear isomorphism $\varphi_{U_i,x}$ preserves the orientations of $p^{-1}(x)$ and the standard orientation of $\mathbb{R}^n$;
(ii) for $x \in U_j \cap U_i$, the linear isomorphism $\varphi_{U_i,x} \circ \varphi^{-1}_{U_j,x}$ preserves the orientation $\mathfrak{o}_x$ of $p^{-1}(x)$.

Then, we say that $\mathfrak{o}_\xi := \{\mathfrak{o}_x\}_{x \in B}$ is an *orientation sheaf* of the vector bundle ξ. A vector bundle ξ is said to be *orientable* if there exists an orientation sheaf of ξ. An orientable vector bundle ξ together with an orientation sheaf $\mathfrak{o}_\xi$ will be called an *oriented vector bundle*. For two vector bundles $\xi := (p, E, B)$ and $\xi' := (p', E', B)$ with orientations sheaves $\mathfrak{o}_\xi = \{\mathfrak{o}_x\}_{x\in B}$ and $\mathfrak{o}_{\xi'} = \{\mathfrak{o}'_x\}_{x\in B}$, respectively, we denote by $\mathfrak{o}_\xi \odot \mathfrak{o}'_{\xi'}$ the orientation sheaf $\{\mathfrak{o}_x \odot \mathfrak{o}'_x\}_{x\in B}$ on $\xi \oplus \xi'$, and we say that the orientation $\mathfrak{o}_\xi \odot \mathfrak{o}'_{\xi'}$ of $\xi \oplus \xi'$ is induced by the orientation $\mathfrak{o}_\xi$ of ξ followed by the orientation $\mathfrak{o}_{\xi'}$ of ξ'.

We say that a manifold M is *orientable* if its tangent vector bundle $\tau(M)$ is orientable. An orientation sheaf $\mathfrak{o}_M := \mathfrak{o}_{\tau(M)}$ of $\tau(M)$ is also called an *orientation* of M. In such a case, we will simply write $(M, \mathfrak{o}_M)$ to indicate that M is considered with the specific orientation $\mathfrak{o}_M$.

Suppose that $(M, \mathfrak{o}_M)$ is an oriented submanifold of an oriented vector space $(V, \mathfrak{o}_V)$. Then, the *normal vector bundle* $\nu(M)$ of M in V has a natural orientation $\mathfrak{o}_\nu$ induced from M and V, which satisfies $\mathfrak{o}_\nu \odot \mathfrak{o}_M = \{\mathfrak{o}_V\}_{x\in M}$. Such an orientation $\mathfrak{o}_\nu$ on $\nu(M)$ is called a *positive orientation* of $\nu(M)$ induced from V.

Assume that $f : M \to N$ is a smooth map between two n-dimensional oriented manifold. If for some $x \in M$, the tangent map $df_x : \tau_x(M) \to \tau_{f(x)}(N)$ is an isomorphism, then we put $\operatorname{sign} df_x = 1$ if df_x preserves the orientations of $\tau_x(M)$ and $\tau_{f(x)}(N)$, and $\operatorname{sign} df_x = -1$ otherwise.

2.1.3 Local Brouwer Degree

Let us recall the standard properties of the (local) Brouwer degree of continuous maps from an oriented n-dimensional manifold to $\mathbb{R}^n$.

Let M be an oriented n-dimensional manifold and $f : M \to \mathbb{R}^n$ a continuous map such that $K := f^{-1}(0)$ is compact. The *local Brouwer degree* of f (with respect to the origin) is the integer $\deg(f, M)$ satisfying the following properties:

(1) (Additivity) Let U_1 and U_2 be two open disjoint subsets of M such that $K \subset U_1 \cup U_2$. Then,

$$\deg(f, M) = \deg(f, U_1) + \deg(f, U_2).$$

(2) (Homotopy Invariance) Let $h : [0,1] \times M \to \mathbb{R}^n$ be a homotopy such that $h^{-1}(0)$ is compact. Then, $\deg(h(0,\cdot), M) = \deg(h(1,\cdot), M)$.

(3) (Normalization) If f is a homeomorphism preserving the orientations of M and $\mathbb{R}^n$ then $\deg(f, M) = 1$. If f reverses the orientations of M and $\mathbb{R}^n$, then $\deg(f, M) = -1$.

(4) (Regular Value Property) If f is a smooth mapping such that 0 is a regular value of f, then

$$\deg(f, M) = \sum_{x \in f^{-1}(0)} \operatorname{sign} df_x,$$

where $\operatorname{sign} df_x$ is 1 if $df_x : \tau_x(M) \to \mathbb{R}^n$ preserves the orientations, and -1 otherwise.

(5) (Excision Property) Let $U \subset M$ be an open set such that $f^{-1}(0) \subset U$, then

$$\deg(f, M) = \deg(f, U).$$

Put $B^n := \{x \in \mathbb{R}^n : |x| < 1\}$, $S^n := \partial B^n$. The local Brouwer degree also satisfies the following important property:

(6) (Hopf Property) Two continuous maps

$$\phi, \psi : (\overline{B^n}, S^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

are homotopic if and only if $\deg(\phi) = \deg(\psi)$.

Remark 2.1.9. In the case M is not orientable, the above degree is not correctly defined. However, the residue mod 2 of the integer is well-defined and can be taken as a definition of the "mod 2 degree" in this case. Observe that the mod 2 degree defined this way satisfies properties (2) and (5). Moreover, properties (1), (3) and (4) are also satisfied being understood in the sense of the algebraic operation taken in $\mathbb{Z}_2$.

2.2 Elements of Equivariant Topology and Representation Theory

2.2.1 Basic Concept in Equivariant Topology

Hereafter, G stands for a compact Lie group. By a *subgroup* of G, we mean a closed subgroup of G. Two subgroups H and K of G are *conjugate* if there exists $g \in G$ such that $K = gHg^{-1}$. Obviously, the conjugation relation is an equivalence relation. The equivalence class of H is called a *conjugacy class* of H in G and will be denoted by (H). We denote by $\Phi(G)$ the set of all the conjugacy classes (H) in G. For two subgroups H and K of G, we write

$$(H) \leq (K), \quad \text{if } H \subset g^{-1}Kg \text{ for some } g \in G. \tag{2.1}$$

The relation $\leq$ defines a partial order on the set $\Phi(G)$. For a subgroup H of G, we use $N(H)$ to denote the *normalizer* of H in G, and $W(H)$ to denote the *Weyl group* $N(H)/H$ in G. Since H is assumed to be closed, $N(H)$ is thus also a closed subgroup of G. Moreover, since H is a closed normal subgroup in $N(H)$, hence $W(H)$ is a compact Lie group.

Definition 2.2.1. A *topological transformation group* is a triple (G, X, φ), where X is a Hausdorff topological space and $\varphi : G \times X \to X$ is a continuous map such that:

(i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all g, $h \in G$ and $x \in X$;
(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

The map φ is called a *G-action* on X and the space X, together with a given action φ of G, is called a *G-space*. Similarly, one can define the *right G-action* and call X a *space-G* (sometimes also called *right G-space*). We shall use the notation gx, for $\varphi(g, x)$, and xg in the case of a space-G. For $K \subset G$ and $A \subset X$, we put $K(A) := \{gx : g \in K, \ \ x \in A\}$ and for $g \in G$ we write $gA := \{gx : x \in A\}$. A set $A \subset X$ is said to be *G-invariant*, if $G(A) = A$. Notice that if A is a compact set, $G(A)$ is also compact. Observe that on any Hausdorff topological space X, one can define the *trivial action* of G by $gx = x$ for all $g \in G$ and $x \in X$.

Let X be a G-space. For $x \in X$, denote by $G_x := \{g \in G : \ gx = x\}$ the *isotropy group* of x and by $G(x) := \{gx \in X \ : \ g \in G$ the *orbit* of x. A G-action is called *free* on X, if $G_x = \{e\}$ for all $x \in X$. The conjugacy class (G_x) will be called the *orbit type* of x. The symbol $\Phi(G; X)$ stands for the set of all orbit types occuring in X. For an invariant subset $A \subset X$ and a subgroup H of G we put $A^H := \{x \in A : \ G_x \supset H\}$, $A_H := \{x \in A : \ G_x = H\}$, $A_{(H)} := \{x \in A : \ (G_x) = (H)\}$. By direct verification, A^H is $N(H)$-invariant, as well as $W(H)$-invariant. Moreover, the $W(H)$-action on A_H is free.

For a G-space X, consider an equivalence relation $\sim$ on X: $x \sim y$ if and only if $y = gx$ for some $g \in G$. Denote by X/G the quotient set $X/\sim$. Then, X/G endowed with the quotient topology is called the *orbit space* of X. For a right G-space X, the orbit space will be denoted by $G\backslash X$.

Let G_1 and G_2 be compact Lie groups and assume X to be a G_1-space and space-G_2 such that $(g_1x)g_2 = g_1(xg_2)$ for all $g_i \in G_i$, $i = 1, 2$, $x \in X$. In this case, we call X a *G_1-space-G_2*, and the orbit space is denoted by $G_2\backslash X/G_1$.

In particular, a subgroup H (resp. L) of G acts on G by the left (resp. the right) G-action, so G can be viewed as an H-space (resp. space-L). The corresponding orbit space G/H (resp. $L\backslash G$) is canonically identified with the set of left cosets $\{gH\ :\ g \in G\}$ (resp. the set of right cosets $\{Hg\ :\ g \in G\}$). By the associativity of G, G also becomes an H-space-L, with its orbit space $L\backslash G/H$ being identified with the set of double cosets.

Definition 2.2.2. For two G-spaces X and Y, a continuous map $f : X \to Y$ is called a *G-equivariant* map, or simply a *G-map*, if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$.

For more details on the equivariant topology, we refer to [25, 47, 104].

2.2.2 Representation of Compact Lie Groups

Representations of a compact Lie group G are examples of G-spaces which are of particular interest for us.

Finite-dimensional G-Representations

Definition 2.2.3. A finite-dimensional real (resp. complex) vector space V is called *a real (resp. complex) G-representation*, if V is a G-space such that the translation map $T_g : V \to V$, defined by $T_g(v) := gv$ for $v \in V$, is an $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operator for every $g \in G$. An inner product (resp. Hermitian inner product) $\langle\cdot,\cdot\rangle : V \oplus V \to \mathbb{R}$ (resp. $\langle\cdot,\cdot\rangle : V \oplus V \to \mathbb{C}$) is called *$G$-invariant*, if $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$, u, $v \in W$. A G-representation together with a G-invariant inner product is called an *orthogonal (resp. unitary) G-representation*.

A G-invariant linear subspace $\widetilde{V} \subset V$ is called a *G-subrepresentation* of V. Two representations V_1 and V_2 are called *equivalent* or *isomorphic*, if there is an G-equivariant isomorphism $A : V_1 \rightarrowtail V_2$, and we write $V_1 \cong V_2$. We say that V is an *irreducible G-representation*, if it has no subrepresentation different from $\{0\}$ and V. Otherwise, V is called *reducible*.

Given a G-representation V, the map $T : G \to GL(W)$, $T(g) := T_g$, is a continuous homomorphism, which is in fact an analytic map (cf. [142]). Based on the usage of the Haar integral for a compact Lie group, it can be proved that every real (resp. complex) G-representation is equivalent to an orthogonal (resp. unitary) representation $T : G \to O(n)$ (resp. $T : G \to U(n)$).

For two G-representations V_1 and V_2, denote by $L^G(V_1, V_2)$ the space of all linear G-equivariant maps $A : V_1 \to V_2$, and by $GL^G(V_1, V_2)$ its subspace of all G-equivariant isomorphisms. Put $L^G(V) := L^G(V, V)$ and $GL^G(V) := GL^G(V, V)$.

In the case of two irreducible G-representations $\mathcal{V}_1$ and $\mathcal{V}_2$, Schur's Lemma states that every equivariant linear map $A : \mathcal{V}_1 \to \mathcal{V}_2$ is either an isomorphism or zero. It follows that every complex irreducible G-representation $\mathcal{U}$ is ***absolutely irreducible***, i.e. every equivariant linear map $A : \mathcal{U} \to \mathcal{U}$ satisfies $A = \lambda \mathrm{Id}$, for some $\lambda \in \mathbb{C}$. Consequently, we have that $\dim_{\mathbb{C}} L^G(\mathcal{U}^1, \mathcal{U}^2) = 1$ or 0 (where $\mathcal{U}^1$ and $\mathcal{U}^2$ are two complex G-representations). Using this fact, it can be easily proved that every complex irreducible G-representation of an abelian compact Lie group G is one-dimensional. In the case $\mathcal{V}$ is a *real* irreducible G-representation, the set $L^G(\mathcal{V})$ is a finite-dimensional associative division algebra over $\mathbb{R}$, so it is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and we call $\mathcal{V}$ to be of *real, complex* or *quaternionic type*, respectively.

Characters of G-representations

For a finite-dimensional real (resp. complex) G-representation W, with the corresponding homomorphism $T : G \to GL(W)$, the *character* of W is the function $\chi_W : G \to \mathbb{R}$ (resp. $\chi_W : G \to \mathbb{C}$), defined by

$$\chi_W(g) = \mathrm{Tr}\,(T(g)), \quad g \in G,$$

where Tr stands for the trace of the representing matrix.

The character is a class function, which takes a constant value on a fixed conjugacy class. It carries the essential information about the representation. For example, a real or complex representation is determined up to isomorphism by its character. Also, if a representation is the direct sum of subrepresentations, then the corresponding character is the sum of the characters of those subrepresentations (cf. [27]).

The characters of G-representations are mainly used in Appendix A2 to distinguish different irreducible representations of G used in this thesis.

Convention of Notations

We use the letter V to denote a real G-representation, while the letter U is reserved for complex G-representations. In the case the type of a G-representation is not specified, we apply the letter W. By the completeness theorem of Peter-Weyl, a compact Lie group G has only countably many irreducible G-representations (cf. [27]), so we assume that a complete *catalogue*, indexed by numbers $n = 0, 1, 2, 3, \dots$, of these irreducible representation is available. In Appendix A2, we describe several such catalogues for the groups used in this thesis. In the case of real G-representations, we denote them by $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2, \dots$ (where $\mathcal{V}_0$ always stands for the trivial irreducible G-representation), and in the case of complex G-representations, by $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\dots$(where $\mathcal{U}_0$ is the trivial complex irreducible G-representation), and in the case the type of an irreducible G-representation is not clearly specified as real or complex, we denote them by $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\dots$(where again $\mathcal{W}_0$ is the trivial irreducible G-representation).

Remark 2.2.4. In a special case $G = \Gamma \times S^1$ for a compact Lie group Γ, notice that every complex irreducible Γ-representation $\mathcal{U}_j$ can be converted to an real irreducible $\Gamma \times S^1$-representation by

$$(\gamma, z)w = z^l \cdot (\gamma w), \quad (\gamma, z) \in \Gamma \times S^1, \quad w \in \mathcal{U}_j, \tag{2.2}$$

where '·' denotes complex multiplication. We denote by $\mathcal{V}_{j,l}$ the real $\Gamma \times S^1$-representation obtained in this way.

A summary of our convention is presented in Table 2.1. Exceptional notations will be applied to the irreducible S^1-representations. We denote by ${}^l\mathcal{U}$, $l = 0, \pm 1, \pm 2, \dots$, the complex S^1-irreducible representation with the S^1-action given by

$$zw = z^l \cdot w, \quad z \in S^1, \quad w \in {}^l\mathcal{U}, \tag{2.3}$$

	Real	Complex	Unspecified
G-representation	V, $\mathfrak{V}$	U, $\mathfrak{U}$	W, $\mathfrak{W}$
Irreducible G-representation	$\mathcal{V}$	$\mathcal{U}$	$\mathcal{W}$
List of all irreducible G-representations	$\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$	$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ ($\{\mathcal{V}_{j,l}\}$, if $G = \Gamma \times S^1$)	$\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \dots$

Table 2.1. Notational convention for real and complex G-representations

where '·' is the complex multiplication. Similarly, for real irreducible S^1-representations, we will use the notation ${}^l\mathcal{V}$, $l = 0, \pm 1, \pm 2, \dots$.

For a real vector space V, we denote by $V^c := \mathbb{C} \otimes_{\mathbb{R}} V$ the *complexification* of V. Assume that V is a real G-representation. Then, V^c has a natural structure of a complex G-representation defined by $g(z \otimes v) = z \otimes gv$, $z \in \mathbb{C}$, $v \in V$. It is also known that for a real irreducible G-representation V, the complex G-representation V^c is irreducible if and only if V is of real type. Otherwise, if V has a natural complex structure, then V^c, as a complex G-representation, is equivalent to $V \oplus \overline{V}$, where $\overline{V}$ is the conjugate representation of V. In this case V is equivalent to $\overline{V}$ as a complex G-representation, if and only if V is of quaternionic type (cf. [27]).

Isotypical Decompositions

By the complete reducibility theorem, every finite-dimensional G-representation V is a direct sum of irreducible subrepresentations of V, i.e.

$$V = \mathcal{V}^1 \oplus \mathcal{V}^2 \oplus \cdots \oplus \mathcal{V}^m \tag{2.4}$$

where $\mathcal{V}^i$ is an irreducible subrepresentation of V and some of $\mathcal{V}^i$'s may be equivalent. This direct decomposition is not geometrically unique and only defined up to isomorphism.

Among these irreducible subrepresentations, there may be distinct (non-equivalent) subrepresentations, which we denote by $\mathcal{V}_{k_1}, \dots, \mathcal{V}_{k_r}$, including possibly the trivial one-dimensional representation $\mathcal{V}_0$. Let V_{k_j} be the sum of all irreducible subspaces $\mathcal{V}^i \subset V$ equivalent to $\mathcal{V}_{k_j}$. Then,

$$V = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_n}, \tag{2.5}$$

which is called the *isotypical decomposition* of V. In contrast to (2.4), the isotypical decomposition (2.5) is unique. The subspace V_{k_j} is called the *isotypical component* of type $\mathcal{V}_{k_j}$ (or modeled on $\mathcal{V}_{k_j}$).

It will be also convenient to write the isotypical decomposition (2.5) in the form

$$V = V_0 \oplus \cdots \oplus V_r, \tag{2.6}$$

where each isotypical component V_i is modeled on $\mathcal{V}_i$, according to a complete list of irreducible G-representations $\{\mathcal{V}_i\}$. In particular, some V_j in (2.6) may be a trivial subspace.

In the case of a finite-dimensional complex G-representation U, a similar *complex isotypical decomposition* of U can be constructed, namely

$$U = U_0 \oplus U_1 \oplus \cdots \oplus U_s,$$

where the isotypical component U_j is modeled on the complex irreducible G-representation $\mathcal{U}_j$, according to a complete list of complex irreducible G-representations $\{\mathcal{U}_j\}$.

Isotypical Decomposition of $GL^G(V)$

Let V be an orthogonal G-representation and let $GL^G(V)$ be the group of all equivariant linear invertible operators on V. We have the following standard algebraic facts on a decomposition of $GL^G(V)$.

Proposition 2.2.5. (cf. [106]) *Consider the G-isotypical decomposition*

$$V = V_{k_1} \oplus \cdots \oplus V_{k_r}, \tag{2.7}$$

where a component V_{k_i} is modeled on an irreducible representation $\mathcal{V}_{k_i}$. Then,

(i) $GL^G(V) = \bigoplus_{i=1}^{r} GL^G(V_{k_i})$;
(ii) for any isotypical component V_{k_i} from (2.7), we have $GL^G(V_{k_i}) \simeq GL(m, \mathbb{F})$, where $m = \dim V_{k_i}/\dim \mathcal{V}_{k_i}$ and $\mathbb{F} \simeq GL^G(\mathcal{V}_{k_i})$, i.e. $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, depending on the type of the irreducible representation $\mathcal{V}_{k_i}$.

Banach G-Representations

Definition 2.2.6. A real (resp. complex) Banach space W is a *real (resp. complex) Banach G-representation*, if W is additionally a G-space such that the translation map $T_g : W \to W$, defined by $T_g(w) = gw$ for $w \in W$, is a bounded $\mathbb{R}$-linear (resp. bounded $\mathbb{C}$-linear) operator for every $g \in G$. A Banach G-representation W is called *isometric*, if for each $g \in G$, $T_g : W \to W$ is an isometry, i.e. $\|T_g w\| = \|w\|$ for all $w \in W$. The norm $\|\cdot\|$ is called a *G-invariant norm.*

A closed G-invariant linear subspace of W is called a *Banach G-subrepresentation.* Two representations W_1 and W_2 are called *equivalent* or *isomorphic*, if there is an G-equivariant isomorphism $A : W_1 \to W_2$. We say that W is an *irreducible Banach G-representation*, if it contains no G-subrepresentation different from $\{0\}$ and W. Otherwise, W is called *reducible.*

If W is a real (resp. complex) Hilbert space, the inner product (resp. Hermitian inner product) $\langle \cdot, \cdot \rangle$ on W is called *G-invariant*, if $\langle gv, gw \rangle = \langle v, w \rangle$, for all $g \in G$, $v, w \in W$. In this case, W is called an *isometric Hilbert (resp. unitary Hilbert) G-representation.*

For a Banach G-representation W and $r > 0$, denote by

$$B_r(W) := \{w \in W \ : \ \|w\| < r\}.$$

Clearly, all the finite-dimensional G-representations are examples of Banach G-representations. Based on the usage of the Haar integral for G, it can be proved that for every Banach G-representation W, it is possible to construct a G-invariant norm on W equivalent to the initial one.

By the completeness theorem of Peter-Weyl, there exists at most countably many irreducible Banach G-representations of a compact Lie group G. It is also important to notice that all the irreducible Banach G-representations are finite-dimensional (see [106, 116]).

Consider a complete list of all irreducible Banach G-representations, denoted by $\{\mathcal{W}_k\}_{k=0}^{\infty}$. Let W be an isometric Banach G-representation. Then, every irreducible Banach G-subrepresentation of W is equivalent to $\mathcal{W}_k$ for some k. Moreover, there exists a closed G-invariant subspace W_k, called the *isotypical component* of W corresponding to $\mathcal{W}_k$, in which every irreducible subrepresentation of type $\mathcal{W}_k$ is contained (cf. [15]). Define the subspace

$$W_\infty := \bigoplus_k W_k \tag{2.8}$$

which is clearly dense in W. Consequently, W admits the following *isotypical decomposition*

$$W = \overline{\bigoplus_k W_k}. \tag{2.9}$$

In particular, for every G-equivariant linear operator $A : W \to W$, we have that $A(W_k) \subseteq W_k$ for all $k = 0, 1, 2, \ldots$.

We have the following result

Proposition 2.2.7. *(cf. [15]) Given (2.8) and (2.9), for any finite subset $X \subset W_\infty$ the subspace* $\operatorname{span} G(X)$ *spanned by the orbits of points from X, is finite-dimensional and G-invariant.*

For more information on Banach representations we refer to [106, 20, 116].

2.2.3 G-Manifolds

Definition 2.2.8. A finite-dimensional smooth manifold M is a *G-manifold*, if it is a G-space such that the G-action on M is a smooth map.

A vector bundle (p, E, B) is a *smooth G-vector bundle*, if E and B are G-manifolds and $p : E \to B$ is an equivariant smooth mapping admitting smooth local trivializations, as well as the map $g : p^{-1}(x) \to p^{-1}(gx)$ given by $y \mapsto gy$, is an isomorphism of Banach spaces, for all $g \in G$.

For a G-manifold M, the tangent bundle $\tau(M)$ of M is a smooth G-vector bundle. Let W be a Riemannian G-manifold, i.e. W has G-invariant Riemannian metric $\langle \cdot, \cdot \rangle : \tau(W) \times \tau(W) \to \mathbb{R}$. Suppose that M is a G-submanifold of W. Then, the normal vector bundle $\nu(M)$ of M in W is also a smooth G-vector bundle.

Definition 2.2.9. Let H be a closed subgroup of G and let A be an H-space. Define an H-action on $G \times A$ by $\varphi : H \times (G \times A) \to (G \times A)$ with $\varphi(h, (g, a)) = (gh^{-1}, ha)$, for $h \in H$, $g \in G$ and $a \in A$. The orbit space

$$G \underset{H}{\times} A := (G \times A)/H$$

is called the *twisted product* of G and A.

For the twisted product $G \underset{H}{\times} A$, we denote by $[g, a]$ the H-orbit of (g, a). Observe that $G \underset{H}{\times} A$ is a G-space with the G-action $\tilde{\varphi} : G \times (G \underset{H}{\times} A) \to G \underset{H}{\times} A$ defined by $\tilde{\varphi}(g', [g, a]) = [g'g, a]$. By direct verification, we have that

(i) $(G \underset{H}{\times} A)/G$ is homeomorphic to A/H;
(ii) If A is also G-space, then $G \underset{H}{\times} A$ is G-homeomorphic to A.

Given a G-manifold, the following theorem describes the conditions of neighborhoods of each orbit, which is fundamental in the study of the structure of G-manifolds.

Theorem 2.2.10. (Slice Theorem) *(cf. [104]) Let G be a compact Lie group and M a G-manifold. For any $x \in M$, the orbit $G(x)$ is a G-invariant submanifold of M. Let ν denote the normal G-vector bundle of $G(x)$ in M. Then* *the fibre ν_x over x of ν is a representation space of the isotropy group G_x so that ν is isomorphic to*

$$G \underset{G_x}{\times} \nu_x \to G/G_x$$

as smooth G-vector bundles. Moreover, there exist a G-invariant open neighborhood U of $G(x)$ in M and a G-diffeomorphism $f : G \underset{G_x}{\times} \nu_x \to U$ such that the restriction of f to the zero cross-section gives the G-diffeomorphism from G/G_x to $G(x)$ defined by $gG_x \mapsto gx$.

Definition 2.2.11. Let M be a G-manifold. The image $f(\nu_x)$ of ν_x under the G-diffeomorphism f above is called a *slice* of $G(x)$ at x, the representation ν_x of G_x is called a *slice representation*, and U is called a *tubular neighborhood around the orbit* $G(x)$.

Theorem 2.2.12. ([cf. [104],[25]) *Let M be a G-manifold and H a subgroup of G. Then,*

(i) $M_{(H)}$ is a G-invariant submanifold of M;
(ii) $M_{(H)}/G$ is a manifold. If $M_{(H)}$ is connected, then $M_{(H)}/G$ is also connected;
(iii) If (H) is a maximal orbit type in M, then $M_{(H)}$ is closed in M;
(iv) If (H) is a minimal orbit type in M and M/G is connected, then $M_{(H)}/G$ is a connected, open and dense subset of M/G;
(v) M_H is a $W(H)$-invariant manifold with free $W(H)$-action,

where the minimal and maximal orbit types are taken with respect to the partial order relation (2.1).

Corollary 2.2.13. *Let V be a finite-dimensional G-representation. Then, for every orbit type (H) in V, the set $V_{(H)}$ is an invariant submanifold of V. Moreover, the set V_H is an open $W(H)$-invariant dense subset of V^H.*

2.2.4 Bi-Orientability of a Compact Lie Group

For a finite-dimensional smooth orientable G-manifold M, we say that M admits a *G-invariant orientation*, if the G-action preserves an orientation of $\tau(M)$.

It is easy to see that every compact Lie group G, considered as a G-manifold with the G-action defined by left translations (resp. right translations), admits a G-invariant orientation. In this case, we call this G-invariant orientation a *left-invariant orientation* (resp. *right-invariant orientation*) on G.

Definition 2.2.14. (cf. [147, 72])Let G be a compact Lie group. If G admits an orientation which is both left-invariant and right-invariant, then G is said to be *bi-orientable*.

Remark 2.2.15. (cf. [15]) The concept of bi-orientability is closely related to the following problem: *given a free G-manifold M and $x \in M$, does the orbit $N := G(x)$ admit a natural G-invariant orientation?* Since G acts freely on M, there exists a G-diffeomorphism $\mu_x : G \to N$ given by $\mu_x(g) := gx$, $g \in G$, for a certain fixed point $x \in N$. Then, the G-diffeomorphism naturally induces an orientation $\mathfrak{o}_N$ on N from the orientation $\mathfrak{o}_G$ of G. By direct verification, in order for this choice of orientation being independent of the choice of x, one needs to require the orientation $\mathfrak{o}_G$ being invariant with respect to *right* translations of G. On the other hand, the constructed orientation $\mathfrak{o}_N$ of the orbit N is G-invariant, if and only if $\mathfrak{o}_G$ is invariant with respect to *left* translations of G. Consequently, an orbit $G(x) \subset M$ admits a natural G-invariant orientation, if and only if G is bi-orientable (see [147] for more details) .

Examples of bi-orientable compact Lie groups are abelian groups, finite groups or those which have an odd number of connected components (in particular, if G is connected) (cf. [147]). The importance of the notion of bi-orientability rests on the following:

Proposition 2.2.16. (cf. [147]). *Let M be a free smooth finite-dimensional G-manifold and let M/G be connected. Assume M admits a G-invariant orientation. Let M_o be a (fixed) connected component of M and put $G_o := \{g \in G : gM_o = M_o\}$. Then, M_o/G_o is diffeomorphic to M/G as smooth manifolds. Moreover, M_o/G_o is orientable if and only if G_o is bi-orientable.*

Consequently, under the assumptions of Proposition 2.2.16, if G_o is bi-orientable, then there exists an orientation on M/G in a canonical way.

Definition 2.2.17. Let X be a smooth finite-dimensional G-manifold. Assume that $(H) \in \Phi(G; X)$ is such that $W(H)$ is bi-orientable, and X^H admits a

$W(H)$-invariant orientation, denoted by $\mathfrak{o}_X$. For $x \in X^H$, choose a natural orientation $\mathfrak{o}_W$ of the orbit $W(H)(x) \subset X^H$ (cf. Remark 2.2.15). Denote by S_x a slice of $W(H)(x)$ at x in X^H (cf. Definition 2.2.11). An orientation $\mathfrak{o}_S$ on S_x is called *positive*, if $\mathfrak{o}_S$ followed by $\mathfrak{o}_W$ gives the initial orientation $\mathfrak{o}_X$ of X^H. In this case, the slice S_x is called a *positively oriented slice*. Otherwise, the slice will be called a *negatively oriented* slice.

Let V be an orthogonal G-representation. Consider another orthogonal G-representation $\mathbb{R}^k \oplus V$, where G acts trivially on $\mathbb{R}^k$, for $k \geq 0$. We will adopt several notations: $\Phi_k(G)$ stands for the set of all conjugacy classes (H) in G such that $\dim W(H) = k$; $\Phi_k(G, V)$ denotes the set of all orbit types (H) in $\mathbb{R}^k \oplus V$ such that $(H) \in \Phi_k(G)$; $\Phi_n^+(G) \subset \Phi_n(G)$ stands for the set of all conjugacy classes (H) such that $W(H)$ is bi-orientable; $\Phi_n^+(G, V) \subset \Phi_n(G, V)$ denotes the set of all orbit types (H) in $\mathbb{R}^n \oplus V$ such that $(H) \in \Phi_n^+(G)$; $A_n^+(G)$ stands for the free $\mathbb{Z}$-module generated by $\Phi_n^+(G)$.

2.3 Regular Normal Approximations

Let V be an orthogonal G-representation, and $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded G-invariant subset (where $n \geq 0$ and G acts trivially on $\mathbb{R}^n$).

Definition 2.3.1. A continuous G-equivariant map $f : \mathbb{R}^n \oplus V \to V$ (resp. a pair (f, Ω)) is called *Ω-admissible* (resp. *an admissible pair*), if $f(x) \neq 0$ for all $x \in \partial\Omega$. An equivariant homotopy $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is called *Ω-admissible*, if $h_t := h(t, \cdot)$ is Ω-admissible for all $t \in [0,1]$.

Many theoretical problems of the equivariant homotopy classification of Ω-admissible maps relate to the questions of how to separate zeros of different orbit types, and how to choose representatives of equivariant homotopy classes admitting reasonable transversality conditions. The first question gives rise to the so-called *normality condition*, while the second one is more delicate, as the equivariance "gets in conflict" with regularity. Therefore, one seeks for special transversality requirements that are compatible with our considerations (for a general discussion related to different G-actions on a domain and target, we refer to [19, 101, 120]).

Definition 2.3.2. (cf. [72, 119, 120]). Let V be an orthogonal G-representation, $\Omega \subset \mathbb{R}^n \oplus V$ an open bounded G-invariant subset and $f : \mathbb{R}^n \oplus V \to V$ an

Ω-admissible G-equivariant map. We say that f is *normal* in Ω, if for every $\alpha := (H) \in \Phi(G;\Omega)$ and every $x \in f^{-1}(0) \cap \Omega_H$, the following *$\alpha$-normality condition* at x is satisfied: there exists $\delta_x > 0$ such that for all $w \in \nu_x(\Omega_\alpha)$ with $\|w\| < \delta_x$,

$$f(x+w) = f(x) + w = w.$$

Similarly, an Ω-admissible G-homotopy $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is called a *normal homotopy* in Ω, if for every $\alpha := (H) \in \Phi(G;\Omega)$ and for every $(t,x) \in h^{-1}(0) \cap ([0,1] \times \Omega_H)$, the following *$\alpha$-normality* condition at (t,x) is satisfied: *There exists $\delta_{(t,x)} > 0$ such that for all $w \in \nu_{(t,x)}([0,1] \times \Omega_\alpha)$ with* $\|w\| < \delta_{(t,x)}$,

$$h(t, x+w) = h(t,x) + w = w.$$

Definition 2.3.3. (cf. [72, 119, 120]). Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded invariant set and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. We say that f is a *regular normal* map in Ω if

(i) f is of class C^1;
(ii) f is normal in Ω;
(iii) for every $(H) \in \Phi(G;\Omega)$, zero is a regular value of

$$f_H := f|_{\Omega_H} : \Omega_H \to V^H.$$

Similarly, one can define a *regular normal* homotopy in Ω. The importance of regular normal maps is outlined in the following propositions.

Proposition 2.3.4. (cf. [8], [120]) *Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded invariant set, and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map being regular and normal. Then for every $x \in f^{-1}(0) \cap \Omega$ we have* $\dim(W(G_x)) \le n$.

Proposition 2.3.5. (cf. [119], also see [120, 135, 187]). *Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded invariant set and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. Then for every $\eta > 0$ there exists a regular normal (in Ω) G-equivariant map $\tilde{f} : \mathbb{R}^n \oplus V \to V$ such that $\sup_{x\in\Omega} \|\tilde{f}(x) - f(x)\| < \eta$. Similarly, if $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is an Ω-admissible G-equivariant homotopy, then for every $\eta > 0$ there exists a regular normal (in Ω) G-equivariant homotopy $\tilde{h} : [0,1] \times \mathbb{R}^n \oplus V \to V$ such that $\sup_{(t,x)\in[0,1]\times\Omega} \|\tilde{h}(t,x) - h(t,x)\| < \eta$. In addition, if h_0 and h_1 are regular normal in Ω, then $\tilde{h}_0 = h_0$ and $\tilde{h}_1 = h_1$.*

2.4 The Sets $N(L,H)$ and Numbers $n(L,H)$

The sets $N(L,H)$ and numbers $n(L,H)$ play an essential role in several recurrence formulae, based on which the equivariant degrees are computed.

Definition 2.4.1. (cf. [104]) Given two closed subgroups $L \subset H$ of a compact Lie group G, we define the set

$$N(L,H) := \left\{ g \in G : gLg^{-1} \subset H \right\}.$$

and we put

$$n(L,H) := \left| \frac{N(L,H)}{N(H)} \right|, \tag{2.10}$$

where the symbol $|X|$ stands for the cardinality of the set X.

Remark 2.4.2. Since H is closed and the G-action on G itself is smooth, one shows that $N(L,H)$ is a closed subset of G, hence it is a compact set. Moreover, define an H-action on G by $(h,g) \mapsto hg$, for $h \in H$, $g \in G$, then $N(L,H)$ is an H-invariant subset of G. Consider the H-orbit space $N(L,H)/H$.

(i) Define an $N(H)$-action on $N(L,H)/H$ given by $\varphi : N(H) \times N(L,H) \to N(L,H)$, where

$$\varphi(n, Hg) := H(ng), \quad \text{for } n \in N(H),\ g \in N(L,H).$$

By direct verification, the action is well-defined and the kernel of the action coincides with H, meaning that $\varphi(n, Hg) = Hg$ if and only if $n \in H$. Therefore, $N(L,H)/H$ is in fact a (left) $W(H)$-invariant subset of G, and the $W(H)$-action is free.

(ii) Similarly, define an $N(L)$-action on $N(L,H)/H$ given by $\psi : N(L) \times N(L,H) \to N(L,H)$, where

$$\psi(n', Hg) := H(gn'), \quad \text{for } n' \in N(L),\ g \in N(L,H).$$

One verifies that the action is well-defined and L lies in the kernel of the action, meaning that for every $l \in L$, $\psi(l, Hg) = Hg$ for all $g \in N(L,H)$. Consequently, $N(L,H)/H$ is a (right) $W(L)$-invariant subset of G.

On the other hand, consider G/H as an L-space, with the action given by $(l, gH) \mapsto lgH$. Then, the L-fixed-point space $(G/H)^L$ is naturally a (left) $W(L)$-invariant space. The following result is established in [104].

Proposition 2.4.3. *The map* $Ha \mapsto a^{-1}H$ *defines a* $W(L)$*-equivariant homeomorphism from* $N(L,H)/H$ *to* $(G/H)^L$.

Moreover, we have the following (cf. [25])

Proposition 2.4.4. *Let* $L \subset H$ *be two closed subgroups of the compact Lie group* G. *Consider* $(G/H)^L$ *as the left* $W(L)$*-space. Then, the corresponding orbit space* $(G/H)^L/W(L)$ *is finite.*

Based on Proposition 2.4.3 and Proposition 2.4.4, we prove the following

Proposition 2.4.5. *Let* $L \subset H$ *be two closed subgroups of a compact Lie group* G. *Then,*

(i) $\dim W(L) \geq \dim W(H)$;
(ii) let M *be a connected component of the set* $N(L,H)/H$, *then* $\dim W(H) \leq \dim M \leq \dim W(L)$;
(iii) in the case $\dim W(L) = \dim W(H) = k$, *we have the number* $n(L,H)$ *is finite and the set* $N(L,H)/H$ *is a closed* k*-dimensional submanifold of* G/H.

Proof: Since (i) is a direct consequence of (ii), we prove (ii) and (iii) only.

(ii) Combining Proposition 2.4.3 with Proposition 2.4.4, we have that $N(L,H)/H$, when viewed as a right $W(L)$-space, consists of a finite number of $W(L)$-orbits. By the fundamental homomorphism theorem in algebra, each of these $W(L)$-orbits is homeomorphic to $W(L)/L_o$ for some subgroup $L_o \subset W(L)$. As a connected component, M lies in one of these $W(L)$-orbit, as a closed submanifold, with the dimension $\dim(W(L)/L_o)$. Clearly,

$$\dim(W(L)/L_o) \leq \dim W(L).$$

It follows that

$$\dim M \leq \dim W(L).$$

On the other hand, viewed as a right $W(H)$-invariant space, the set $N(L,H)/H$ is a free $W(H)$-space (cf. Remark 2.4.2(i)). Thus, the natural projection

$$p : N(L,H)/H \to (N(L,H)/H)/W(H) \simeq N(L,H)/N(H), \tag{2.11}$$

is a fibre bundle with the fiber $W(H)$. Hence, we have the following dimension relation

$$\dim M = \dim p(M) + \dim W(H) \geq \dim W(H).$$

Therefore, we proved $\dim W(H) \leq \dim M \leq \dim W(L)$.

(iii) In the case $\dim W(L) = \dim W(H) = k$, by (ii), every connected component of $N(L,H)/H$ has the same dimension k, being a submanifold of certain $W(L)$-orbit. Consequently, the set $N(L,H)/H$ is a closed k-dimensional submanifold of G/H, and the fibre bundle (2.11) induces a dimension relation

$$\begin{aligned} k = \dim N(L,H)/H &= \dim N(L,H)/N(H) + \dim W(H) \\ &= \dim N(L,H)/N(H) + k, \end{aligned}$$

which forces $\dim N(L,H)/N(H) = 0$. By the compactness of $N(L,H)$, the orbit space $N(L,H)/N(H)$ is also compact, which proves that the number $n(L,H)$ is finite. □

The number $n(L,H)$ defined for two closed subgroups of G with $\dim W(H) = \dim W(L)$ has a very simple geometric interpretation.

Lemma 2.4.6. *Let L and H be two closed subgroups of a compact Lie group G such that $L \subset H$ and $\dim W(L) = \dim W(H)$. Then $n(L,H)$ represents the number of different subgroups $\widetilde{H}$ in the conjugacy class (H) such that $L \subset \widetilde{H}$. In particular, if V is an orthogonal G-representation such that (L), $(H) \in \Phi(G;V)$, $L \subset H$, then $V^L \cap V_{(H)}$ is a disjoint union of exactly $m = n(L,H)$ sets of V_{H_j}, $j = 1, 2, \dots, m$, satisfying $(H_j) = (H)$.*

Proof: Notice that $N(L,H)$ can be rewritten as

$$N(L,H) = \{g \in G : L \subset gHg^{-1}\}.$$

Define $\mathcal{H} := \{gHg^{-1} \ : \ g \in G,\ L \subset gHg^{-1}\}$ and a map $b : N(L,H) \to \mathcal{H}$ by $b(g) = gHg^{-1}$, for $g \in N(L,H)$. Consider $N(L,H)$ as a left $N(H)$-space (cf. Remark 2.4.2(i)). By direct verification, b is constant on each $N(H)$-orbit. Thus, there exists a natural factorization $\bar{b} : N(L,H)/N(H) \to \mathcal{H}$ of b. It is then easy to check that $\bar{b}$ is one-to-one and onto. By Proposition 2.4.5, the set $N(L,H)/N(H)$ is a finite set of order $n(L,H)$. Therefore, by the bijection $\bar{b}$, $n(L,H)$ also represents the order of $\mathcal{H}$, i.e. the number of different subgroups $\widetilde{H}$ in the conjugacy class (H) such that $L \subset \widetilde{H}$.

Assume now that V is an orthogonal G-representation, (L), $(H) \in \Phi(G;V)$, and $L \subset H$. Then, $V_H \subset V^L$. Moreover, $gV_H \subset V^L$ if and only if $g \in N(L,H)$. On the other hand, $gV_H = V_H$ if and only if $g \in N(H)$. Therefore, the conclusion follows. □

Remark 2.4.7. For $g_a, g_b \in G$, consider the two subsets $N(L,H)$ and $N(\widetilde{L},\widetilde{H})$ of G, where $\widetilde{L} := g_a L g_a^{-1}$, $\widetilde{H} := g_b H g_b^{-1}$. Define a map $f : N(L,H) \to N(\widetilde{L},\widetilde{H})$ by $f(g) := g_b g g_a^{-1}$. It is easy to check that f is well-defined and it provides a homeomorphism between $N(L,H)$ and $N(\widetilde{L},\widetilde{H})$. Furthermore, consider $N(L,H)$ as a left $N(H)$-space and $N(\widetilde{L},\widetilde{H})$ as a left $N(\widetilde{H})$-space (cf. Remark 2.4.2(i)). Then, f actually factorizes through the orbit spaces, as indicated by a commutative diagram shown in Figure 2.4.7, where we used the fact $N(\widetilde{H}) = g_b N(H) g_b^{-1}$. In particular, $\bar{f}$ provides a homeomorphism between $N(L,H)/N(H)$ and $N(\widetilde{L},\widetilde{H})/N(\widetilde{H})$.

$$\begin{tikzcd}
N(L,H) \arrow[r, "f"] \arrow[d, "p"] & N(\widetilde{L},\widetilde{H}) \arrow[d, "\widetilde{p}"] \\
N(L,H)/N(H) \arrow[r, "\overline{f}"] & N(\widetilde{L},\widetilde{H})/N(\widetilde{H})
\end{tikzcd}$$

Fig. 2.1. Factorization through the orbit spaces.

By Remark 2.4.7, whenever $N(L,H) \neq \emptyset$ (or equivalently, $n(L,H) \neq 0$), we can always choose L and H from the conjugacy classes (L) and (H), such that $L \subset H$. In the case, this is not possible, it simply implies that $N(L,H) = \emptyset$.

Given subgroups $L \subset H \subset G$, consider the H-orbit space $N(L,H)/H$ (cf. Remark 2.4.2). By the compactness of $N(L,H)/H$, it has only a finite number of connected components, denoted by M_i, $i = 1, 2, \ldots, k$. Put

$$\operatorname{Dim} N(L,H)/H := \max\{\dim M_i : i = 1, 2, \ldots, k\}.$$

Lemma 2.4.8. *Assume that $L' \subset L \subset H$ are three subgroups of G. Then,*

$$\mathrm{Dim\, N(L,H)/H \leq Dim\, N(L',H)/H}.$$

Proof: Notice that $N(L,H) \subset N(L',H)$, therefore

$$\frac{N(L,H)}{H} \subset \frac{N(L',H)}{H},$$

and the conclusion follows. □

The numbers $n(L, H)$, whenever are finite, play an important role in the computation of multiplication tables of Burnside rings and the corresponding modules (and, therefore, may be used to establish partial results on the multiplication structure of the Euler ring $U(G)$). However, the main assumption providing the finiteness of $n(L, H)$ is not satisfied for arbitrary $L \subset H \subset G$. Below we introduce a notion close in spirit to $n(L, H)$.

Definition 2.4.9. Given subgroups $L \subset H \subset G$, we say that L is $\mathfrak{N}$*-finite* in H if the space $N(L, H)/H$ is finite. For a given subgroup H, denote by $\mathfrak{N}(H)$ the set of all conjugacy classes (L) such that there exists $\tilde{L} \in (L)$ being $\mathfrak{N}$-finite in H. For $(L) \in \mathfrak{N}(H)$, put

$$m(L, H) := |N(L, H)/H|,$$

where $|X|$ stands for the number of elements in the set X.

Remark 2.4.10. Let $L \subset H \subset G$.

(i) Take a subgroup $L' \subset L$. If L' is $\mathfrak{N}$-finite in H, then L is $\mathfrak{N}$-finite in H (cf. Lemma 2.4.8).

(ii) It follows from Proposition 2.4.5(ii) that, if $W(L)$ is finite, then L is $\mathfrak{N}$-finite.

(iii) Finally, if $W(L)$ and $W(H)$ are finite, then

$$m(L, H) = n(L, H) \cdot |W(H)|.$$

We complete this subsection with the following simple but important observation.

Proposition 2.4.11. *Let $L \subset H \subset G$. Consider the set $N(L, H) \subset G$ as an $N(H)$-space-$N(L)$ (cf. Remark 2.4.2). Then, the corresponding orbit space $N(L)\backslash N(L, H)/N(H)$ is finite, i.e. there exist g_1, g_2, …, $g_k \in G$ such that*

$$N(L, H) = N(H)g_1N(L) \sqcup N(H)g_2N(L) \sqcup \cdots \sqcup N(H)g_kN(L),$$

where $N(H)g_jN(L)$ denotes a double coset, for $j = 1, 2, \ldots, k$, and $\sqcup$ stands for the disjoint union.

Proof: Combining Proposition 2.4.3 with Proposition 2.4.4, we have that $N(L,H)/H$, when viewed as a right $W(L)$-space, consists of a finite number of $W(L)$-orbits. This implies that there exist $g_1, g_2, \dots, g_m \in N(L,H)$ such that

$$\begin{aligned} N(L,H) &= Hg_1W(L) \sqcup Hg_2W(L) \sqcup \cdots \sqcup Hg_mW(L) \\ &\subset N(H)g_1N(L) \cup N(H)g_2N(L) \cup \cdots \cup N(H)g_mN(L) \\ &= N(H)g_{m_1}N(L) \sqcup N(H)g_{m_2}N(L) \sqcup \cdots \sqcup N(H)g_{m_k}N(L), \end{aligned}$$

for some $g_{m_1}, g_{m_2}, \dots, g_{m_k} \in N(L,H)$. ☐

2.5 Fundamental Domains

Definition 2.5.1. Let Q be a topological group and X a finite-dimensional metric Q-space. Let $D_0 \subset X$ be open in its closure D. Then D is a *fundamental domain* of the Q-action on X if the following conditions are satisfied:

(i) $Q(D) = X$;
(ii) $g(D_o) \cap h(D_o) = \emptyset$ for distinct elements $g, h \in Q$;
(iii) $X \setminus Q(D_o) = Q(D \setminus D_o)$;
(iv) $\dim D = \dim X/Q$, $\dim(D \setminus D_o) < \dim D$, $\dim(X \setminus Q(D_o)) < \dim X$ where "dim" is the covering dimension.

Remark 2.5.2. The conditions (i)-(ii) imply that a fundamental domain is a set of representatives of G-orbits, whose interior contains at most one representative from each orbit. The conditions (iii)-(iv) require some compatibility of the fundamental domain and the group action. Notice that the fundamental domain is not necessarily unique, but typically chosen to be a convenient connected part of the space.

Example 2.5.3. (i) Let $Q := \mathbb{Z}_N$ be the cyclic group of order N, generated by ξ, and $X := B_1$ be the unit disk on the complex plane $\mathbb{C}$, where ξ acts as the multiplication by the complex number $e^{i\frac{2\pi}{N}}$, i.e. the rotation by angle $\frac{2\pi}{N}$. In this case, a fundamental domain of $\mathbb{Z}_N$-action on B_1 is a sector of angle $\frac{2\pi}{N}$.

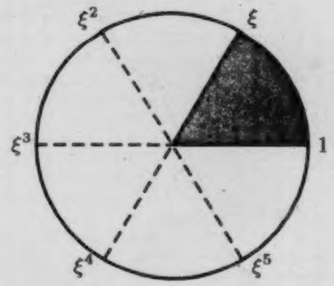

Fig. 2.2. Fundamental domain of the $\mathbb{Z}_N$-action.

(ii) Let $Q := D_N$ be the dihedral group of order $2N$, composed of $\mathbb{Z}_N$ and $\kappa\mathbb{Z}_N$, where $\kappa\xi = -\xi\kappa$. Consider the unit disk $X := B_1 \subset \mathbb{C}$, where ξ acts as rotation and κ acts as the reflection with respect to the real line. In this case, a fundamental domain is a sector of smaller angle $\frac{\pi}{N}$.

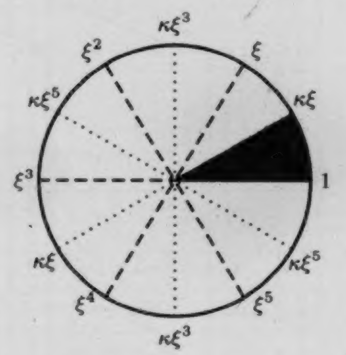

Fig. 2.3. Fundamental domain of the D_N-action.

In fact, a general result about the existence of a fundamental domain is proved in [120]:

Proposition 2.5.4. *Let G be a compact Lie group, and let X be a finite-dimensional metric G-space on which G acts freely. Then a fundamental domain $D \subset X$ always exists.*

Definition 2.5.5. Under the notations of Definition 2.5.1, assume there exists an open *contractible* subset $T_0 \subset X/Q$ such that the natural projection $p : X \to X/Q$ induces the homeomorphism $p|_{D_0} : D_0 \to T_0$. Then D is called a *regular fundamental domain*.

Theorem 2.5.6. *(cf. [12, 15]) Let G be a compact Lie group. For any smooth finite-dimensional free G-manifold X such that X/G is connected, there always exists a regular fundamental domain D.*

Proof: Since every smooth connected manifold admits a (smooth) triangulation (cf. [179], p. 124-135), the proof is essentially based on the following:

Lemma 2.5.7. *Let M be a smooth connected n-dimensional manifold (in general non-compact), and let $\mathcal{S} := \left\{ s_i^k : i \in J^k, \;\; k = 0, 1, 2, \ldots, n \right\}$ be a smooth triangulation of M, where the sets of indices J^k are countable. Then there always exists a subset T_o of M satisfying the following conditions:*

(i) T_o is open in M;
(ii) T_o is dense in M;
(iii) T_o is contractible;
(iv) $M \setminus T_o$ is contained in the $n-1$-dimensional skeleton.

Proof: For a given k-dimensional simplex s^k, we denote by $\mathring{s}^k$ its interior. We call the n-dimensional simplices in $\mathcal{S}$ s_1^n, s_2^n, ... and begin our recursive definition with $T_1 := \mathring{s}_1^n$ and $\mathcal{S}_1 := \mathcal{S} \setminus \{s_1^n\}$.

Assume now that T_m and $\mathcal{S}_m \subset \mathcal{S}$ are already constructed with T_m being open in M and contractible. If $\mathcal{S}_m$ still contains n-dimensional simplices, we chose the minimal $j_{m+1} \in \mathbb{N}$ such that

(a) $s^n_{j_{m+1}} \in \mathcal{S}_m$;
(b) $s^n_{j_{m+1}} \cap \overline{T}_m$ contains an $(n-1)$-dimensional simplex $s^{n-1}_{k_{m+1}} \in \mathcal{S}_m$.

We define $T_{m+1} := T_m \cup \mathring{s}^{n-1}_{k_{m+1}} \cup \mathring{s}^n_{j_{m+1}}$ and $\mathcal{S}_{m+1} := \mathcal{S}_m \setminus \{s^n_{j_{m+1}}, s^{n-1}_{k_{m+1}}\}$. Clearly, T_{m+1} is open in M and contractible.

Let $T_o := \bigcup_m T_m$ and $\mathcal{S}_o := \bigcap_m \mathcal{S}_m$. By construction, T_o is open and (by connectedness of M) dense in M. Also, $S_o = M \setminus T_o$ is a subset of the $n-1$-dimensional skeleton of $\mathcal{S}$.

In order to show that T_o is contractible, notice that T_o is a CW-complex and for every continuous map $\varphi : S^k \to T_o$, $k = 0, 1, 2, \ldots$, the image $\varphi(S^k)$ is compact, so it is entirely contained in some of the contractible sets T_m. Consequently, φ is null-homotopic, hence $\pi_k(T_o) = 0$ for all $k = 0, 1, 2, \ldots$.

Therefore, T_o is contractible (see [165], Cor. 24, Chap. 7, Sec. 6) and Lemma 2.5.7 is proved. □

Continuation of the proof of Theorem 2.5.6.
Let $p : X \to X/G$ be the natural projection. To complete the proof of Theorem 2.5.6, we take the set $T_o \subset M := X/G$ provided by Lemma 2.5.7 and consider the restriction of p over $p^{-1}(T_o)$. The fiber bundle $p : p^{-1}(T_o) \to T_o$, by contractibility of T_o, is trivial. Fix a trivialization $\psi : p^{-1}(T_o) \to G \times T_o$. We put $D_o := \psi^{-1}(\{1\} \times T_o)$. It is clear (cf. [120]) that $D := \overline{D_o}$ is the regular fundamental domain.

The proof of Theorem 2.5.6 is complete. □

2.6 G-ENRs and The Euler Characteristics

In this section, we investigate the relationships among the Euler characteristics of a G-ENR X, of its orbit space X/G, and of its various kinds of fixed point sets X^H.

Definition 2.6.1. (i) A topological space X is called an *ENR (Euclidean Neighborhood Retract)*, if there exist an open subset O of some Euclidean space $\mathbb{R}^n$ and maps $i : X \to O$, $r : O \to X$ such that $ri = \mathrm{Id}$;

(ii) Let G be a compact Lie group. If an ENR X is a G-space, O is a G-invariant open subset of a G-representation $\mathbb{R}^n$, and the maps i and r are G-equivariant, then X is called a *G-ENR*.

A basic theorem of point set topology states that a separable metric space of dimension $\leq n$ can be embedded into $\mathbb{R}^{2n+1}$ (cf. [92]). Hence,

Lemma 2.6.2. *(cf. [47]) A space is an ENR if and only if it is a finite-dimensional, locally compact, separable, and locally contractible metric space.*

For example, every compact manifold, with or without boundary, is an ENR (cf. [87]). In case of G-ENRs, the following results are established in [102, 47].

Proposition 2.6.3. *(cf. [47]) Let X be a G-ENR. Then the orbit space X/G is an ENR.*

Proposition 2.6.4. *(cf. [102]) Let X be a G-space which is separable metric and finite-dimensional. Then, X is a G-ENR if and only if X is locally compact, has a finite number of orbit types, and for every isotropy group $H \subset G$, the fixed point set X^H is an ENR.*

We have direct consequences of Proposition 2.6.4 (cf. [47]).

Corollary 2.6.5. *(i) A finite G-complex X is a G-ENR;*
(ii) A differentiable G-manifold with a finite number of orbit types is a G-ENR.

One of the important properties of (compact) ENR spaces is

Proposition 2.6.6. *(cf. [50, 87]) The singular homology groups $H_*(X)$ of a compact ENR X are finitely generated, i.e. $H_k(X)$ is finitely generated for all k, and $H_k(X) = 0$ for sufficiently large k.*

Consequently, the Euler characteristic of a compact ENR is defined. More precisely, let X be a compact ENR, the *Euler characteristic* $\chi(X)$ is defined as the alternating sum

$$\chi(X) := \sum_{k=0}^{\infty} (-1)^k \mathrm{rank} H_k(X), \tag{2.12}$$

where $H_*(X)$ denotes the singular homology group of X, the "rank" counts the number of free generators of the group, and the sum is essentially finite (cf. Proposition 2.6.6). It is sometimes more convenient to compute $\chi(X)$ by the corresponding singular cohomology ring of X over reals (cf. [165])

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \dim H^k(X). \tag{2.13}$$

In a similar way, one can define the Euler characteristic for a compact ENR pair (X, A), denoted by $\chi(X, A)$, using the relative singular cohomology $H^*(X, A)$. In case of a non-compact ENR X, we define the Euler characteristic $\chi_c(X)$ through the Alexander-Spanier cohomology with compact supports $\bar{H}_c^*(X)$.

The following lemma indicates a relation between the Alexander-Spanier and singular cohomology.

Lemma 2.6.7. *(cf. [165]) If (X, A) is a pair of paracompact Hausdorff spaces being locally contractible, then there exists an isomorphism between the Alexander-Spanier cohomology and singular cohomology, i.e. $\bar{H}^k(X, A) \cong H^k(X, A)$, for all $k \geq 0$, where $\bar{H}^*(X, A)$ stands for the Alexander-Spanier cohomology of (X, A).*

Taking into account of Lemma 2.6.2, we have

Corollary 2.6.8. *If (X, A) is a pair of ENRs, then there exists an isomorphism between the Alexander-Spanier cohomology and singular cohomology, i.e. $\bar{H}^k(X, A) \cong H^k(X, A)$, for all $k \geq 0$.*

Consider the relation between the Alexander-Spanier cohomology with compact supports and the usual Alexander-Spanier cohomology. The followings are established in [165].

Lemma 2.6.9. *(cf. [165]) If X is a compact Hausdorff space and A is closed in X, then there exists an isomorphism between the Alexander-Spanier cohomology with compact supports and the usual Alexander-Spanier cohomology, i.e. $\bar{H}_c^k(X \setminus A) \cong \bar{H}^k(X, A)$, for all $k \geq 0$.*

Corollary 2.6.10. *(cf. [165]) If X is a locally compact Hausdorff space and X^+ is the one-point compactification of X, then there is an isomorphism $\bar{H}_c^k(X) \cong \widetilde{\bar{H}}^k(X^+)$, for all $k \geq 0$, where $\widetilde{\bar{H}}^*(X^+)$ stands for the reduced Alexander-Spanier cohomology of X^+.*

Based on Corollary 2.6.8 and Lemma 2.6.9, we have the following properties of the Euler characteristics of the ENRs.

Lemma 2.6.11. *Let (X, A) be a pair of compact ENRs. Then,*

(i) the Euler characteristic $\chi_c(X \setminus A)$ is correctly defined in the Alexander-Spanier cohomology with compact supports. Moreover,

$$\chi_c(X \setminus A) = \chi(X, A)$$
$$\chi(X) = \chi(X, A) + \chi(A) = \chi_c(X \setminus A) + \chi(A),$$

where $\chi(\cdot)$ denotes the Euler characteristic defined in the singular cohomology.

(ii) (cf. [47]) let $p : (X, A) \to (Y, B)$ be a continuous map between compact ENR's such that $p(X\backslash A) = Y\backslash B$. Suppose that $p : X\backslash A \to Y\backslash B$ is a fibration whose typical fibre F is a compact ENR. Then,

$$\chi(X, A) = \chi(F)\chi(Y, B).$$

Denote by $T^n := S^1 \times S^1 \times \cdots \times S^1$ an n-dimensional torus (for $n > 0$), which is an n-dimensional connected abelian compact Lie group.

Lemma 2.6.12. *Let X be a compact T^n-ENR space for $n > 0$. Then,*

$$\chi(X) = \chi(X^{T^n}).$$

In particular, if $X^{T^n} = \emptyset$, then $\chi(X) = 0$.

Proof: Take a decomposition of X by $X = \bigcup_{(H)\in\Phi(T^n)} X_{(H)}$, where each $X_{(H)}$ is an open T^n-invariant subset of X. Since $X_{(H)}$ is a fibre bundle with fibre T^n/H, $\chi_c(X_{(H)})$ is a multiple of $\chi(T^n/H)$ (cf. Lemma 2.6.11). Thus, by the additivity of χ,

$$\chi(X) = \sum_{(H)} \chi_c(X_{(H)}) = \sum_{(H)} n_H \cdot \chi(T^n/H), \tag{2.14}$$

where $n_H := \chi_c(X_{(H)}/T^n) \in \mathbb{Z}$.

On the other hand, notice that for all $H \subsetneq T^n$, the orbit space T^n/H is diffeomorphic to a connected abelian compact Lie group of dimension at least one. Hence, it is a torus, and thus $\chi(T^n/H) = 0$. Therefore, the essential summand in (2.14) comes from $(H) = (T^n)$. It follows that $\chi(X) = \chi(X_{(T^n)})$. Since T^n is abelian and T^n is the maximal isotropy in X, $\chi(X_{(T^n)}) = \chi(X_{T^n}) = \chi(X^{T^n})$. ☐

Lemma 2.6.13. *Let G be a compact abelian Lie group and X, Y two G-spaces. Denote by Δ the diagonal subgroup in $G \times G$ and consider $X \times Y$ as Δ-space. Define a left G-action on the orbit space $(X \times Y)/\Delta$ by*

$$\begin{aligned} \rho : G \times (X \times Y)/\Delta &\to (X \times Y)/\Delta \\ (g, \Delta(x, y)) &\mapsto \Delta(x, gy). \end{aligned}$$

Then, for $\Delta(x, y) \in (X \times Y)/\Delta$, its isotropy equals to G_xG_y (i.e. the subgroup of G generated by G_x, G_y).

Proof: Notice that since G is abelian, the action ρ is well-defined.

To verify the statement, take $g \in G$ such that $\Delta(x, y) = \Delta(gx, y)$. Observe that viewed as two Δ-orbits, $\Delta(x, y)$ coincides with $\Delta(x, gy)$ if and only if $\Delta(x, y) \cap \Delta(x, gy) \neq \emptyset$, i.e. there exists $g_1, g_2 \in G$ such that $(g_1x, g_1y) = (g_2x, g_2gy)$. This is equivalent to require that $g_1^{-1}g_2 \in G_x$ and $g_1^{-1}g_2g \in G_y$, which implies that $g \in g_2^{-1}g_1G_y \subset G_xG_y$.

On the other hand, if $g \in G_xG_y$, then there exist $g_x \in G_x$, $g_y \in G_y$ such that $g = g_xg_y$. Thus,

$$\Delta(x, gy) = \Delta(x, g_xg_yy) = \Delta(x, g_xy) = \Delta(g_x^{-1}x, y) = \Delta(x, y),$$

i.e. g belongs to the isotropy group of $\Delta(x, y)$. □

Corollary 2.6.14. *Let X, Y be compact T^n-ENRs, and Δ be the diagonal subgroup of $T^n \times T^n$. Assume that $G_xG_y \neq G$, for all $x \in X$, $y \in Y$. Then,*

$$\chi((X \times Y)/\Delta) = 0.$$

Proof: Consider $(X \times Y)/\Delta$ as a left G-space. By Lemma 2.6.13, $((X \times Y)/\Delta)^G = \emptyset$ if and only if $G_xG_y \neq G$, for all $x \in X$, $y \in Y$, which is satisfied by the assumption. Therefore, by Lemma 2.6.12, $\chi((X \times Y)/\Delta) = \chi\left(((X \times Y)/\Delta)^G\right) = \chi(\emptyset) = 0$. □

Corollary 2.6.15. *Let X, Y be compact T^n-ENRs, and Δ be the diagonal subgroup of $T^n \times T^n$. Assume that $\dim G_x + \dim G_y < \dim G$, for all $x \in X$, $y \in Y$. Then,*

$$\chi((X \times Y)/\Delta) = 0.$$

In particular, it holds for $G = S^1$, $X^{S^1} = Y^{S^1} = \emptyset$.

Proof: It is sufficient to observe that $\dim G_x + \dim G_y < \dim G$ implies that $G_xG_y \neq G$. Hence, the statement follows from Corollary 2.6.14. □

Definition 2.6.16. A subgroup $H \subset G$ is said to be *of maximal rank* if H contains a maximal torus T^n of G.

Proposition 2.6.17. *Let $H \subset G$ be a subgroup of G.*

(i) If H is not of maximal rank, then $\chi(G/H) = 0$.

(ii) If H is of maximal rank, then $W_H(T^n)$ is finite and $\chi(G/H) = |W_G(T^n)|/|W_H(T^n)|$. In particular, $\chi(G/T^n) = |W_G(T^n)|$.

Proof:

(i) If H is not of maximal rank, then G/H admits an action of a torus T^k $(0 < k < n)$ without T^k-fixed-points, and the result follows from Lemma 2.6.12.

(ii) Assume H is of maximal rank. Then, for the proof of the finiteness of $W_H(T)$, we refer to [27], Chap IV, Theorem (1.5). Next, we have a fiber bundle $G/T^n \to G/H$ with the fiber H/T^n. Then, by Lemma 2.6.11(ii), $\chi(G/T^n) = \chi(H/T^n) \cdot \chi(G/H)$. On the other hand, by Lemma 2.6.12 and Lemma 2.4.4, we have

$$\chi(H/T^n) = \chi((H/T^n)^{T^n}) = \chi(N_H(T^n)/T^n) = |W_H(T^n)|, \tag{2.15}$$

from which the statement follows.

□

2.7 Completely Continuous and Condensing Fields

2.7.1 Measure of Noncompactness

For a Banach space $\mathbb{E}$, denote by $\mathcal{B}(\mathbb{E})$ the family of all bounded sets in $\mathbb{E}$.

Definition 2.7.1. A function $\mu : \mathcal{B}(\mathbb{E}) \to \mathbb{R}_+ := [0, \infty)$ is called a *measure of noncompactness* if it satisfies the following conditions for $A, B \in \mathcal{B}(\mathbb{E})$

(μ1) $\mu(A) = 0 \iff \overline{A}$ is compact,
(μ2) $\mu(A) = \mu(\overline{A})$,
(μ3) if $A \subset B$ then $\mu(A) \leq \mu(B)$,
(μ4) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(μ5) $\mu(\eta A) = |\eta|\mu(A)$, $\eta \in \mathbb{R}$,
(μ6) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(μ7) $\mu(\operatorname{conv}(A)) = \mu(A)$, where $\operatorname{conv}(A)$ denotes the convex hull of A.

An example of a measure of noncompactness is the so-called *Hausdorff measure of noncompactness.*

Definition 2.7.2. The function $\chi : \mathcal{B}(\mathbb{E}) \to \mathbb{R}_+$, defined for $A \in \mathcal{B}(\mathbb{E})$ by

$$\chi(A) := \inf\{r > 0 : \underset{X \subset \mathbb{E}}{\exists} \; X \text{ is finite and } A \subset X + rB_1(0)\}, \tag{2.16}$$

is call the *Hausdorff measure of noncompactness.*

Proposition 2.7.3. *The function $\chi : \mathcal{B}(\mathbb{E}) \to \mathbb{R}_+$ defined by (2.16) is a measure of noncompactness.*

Proof: We need to verify that χ satisfies the conditions $(\mu 1)$ — $(\mu 7)$. Notice that $A \subset \mathcal{B}(\mathbb{E})$ is *relatively compact* if and only if it is *totally bounded*, i.e.

$$\underset{\varepsilon>0}{\forall} \ \underset{X\subset\mathbb{E}}{\exists} \ X \text{ is finite and } A \subset X + \varepsilon B_1(0).$$

If $X = \{x_1, \dots, x_n\}$, then the set

$$X + \varepsilon B_1(0) = \bigcup_{k=1}^{n} B_\varepsilon(x_k) \supset A$$

is called *ε-net* for A, thus the condition $(\mu 1)$ immediately follows from the definition of $\chi(A)$. The condition $(\mu 3)$ is trivially satisfied, so in order to show $(\mu 2)$ observe that $\chi(A) \leq \chi(\overline{A})$ and we only need to show that $\chi(A) \geq \chi(\overline{A})$. Put $\chi(A) := \alpha$. Then

$$\underset{\varepsilon>0}{\forall} \ \underset{X\subset\mathbb{E}}{\exists} \quad X \text{ is finite and } A \subset X + \left(\alpha + \frac{\varepsilon}{2}\right)B_1(0). \tag{2.17}$$

Since $\overline{A} \subset X + \left(\alpha + \frac{\varepsilon}{2}\right)\overline{B_1(0)} \subset X + (\alpha+\varepsilon)B_1(0)$, it follows from (2.17) that

$$\underset{\varepsilon>0}{\forall} \ \underset{X\subset\mathbb{E}}{\exists} \ X \text{ is finite and } \quad \overline{A} \subset X + (\alpha+\varepsilon)B_1(0),$$

which implies that

$$\underset{\varepsilon>0}{\forall} \ \chi(\overline{A}) \leq \alpha + \varepsilon = \chi(A) + \varepsilon,$$

i.e. $\chi(\overline{A}) \leq \chi(A)$.

To prove $(\mu 5)$, observe that

$$B_{r_1}(0) + B_{r_2}(0) \subset B_{r_1+r_2}(0), \quad r_1, r_2 > 0$$

thus if for some finite sets X' and X''

$$A \subset X' + r_1 B_1(0) \quad \text{and} \quad B \subset X'' + r_2 B_2(0)$$

then

$$A + B \subset X' + X'' + r_1 B_1(0) + r_2 B_1(0) \subset X + (r_1 + r_2)B_1(0),$$

where $X := X_1 + X_2$, and we get ($\mu 6$). To show ($\mu 7$), observe that for two convex sets $C_1, C_2 \subset \mathbb{E}$, we have that $C_1 + C_2$ is convex. Also, since $\operatorname{conv}(A+B)$ is the smallest convex set containing $A + B$, we immediately obtain

$$\operatorname{conv}(A+B) \subset \operatorname{conv}(A) + \operatorname{conv}(B).$$

By ($\mu 3$), $\chi(A) \leq \chi\big(\operatorname{conv}(A)\big)$. Let $\alpha := \chi(A)$, then by using (2.17) we have

$$\underset{\varepsilon>0}{\forall} \; \underset{X \subset \mathbb{E}}{\exists} \; X \text{ is finite and } \operatorname{conv}(A) \subset \operatorname{conv}(X) + \left(x + \frac{\varepsilon}{2}\right) B_1(0).$$

Since X is finite, $\operatorname{conv}(X)$ is compact and by ($\mu 1$) there exists a finite set $X' \subset \mathbb{E}$ such that

$$\operatorname{conv}(X) \subset X' + \frac{\varepsilon}{2} B_1(0),$$

which implies

$$\operatorname{conv}(A) \subset X' + \frac{\varepsilon}{2} B_1(0) + \left(\alpha + \frac{\varepsilon}{2}\right) B_1(0) \subset X' + (\alpha + \varepsilon) B_1(0),$$

thus

$$\underset{\varepsilon>0}{\forall} \; \chi\big(\operatorname{conv}(A)\big) \leq \alpha + \varepsilon = \chi(A) + \varepsilon$$

i.e. $\chi\big(\operatorname{conv}(A)\big) \leq \chi(A)$ and ($\mu 7$) follows. The proofs of ($\mu 4$) and ($\mu 5$) are straightforward. □

Proposition 2.7.4. *Let $\mathbb{E}$ be a Banach space and $B := B_1(0)$ the unit ball in $\mathbb{E}$, and μ a measure of noncompactness on $\mathcal{B}(\mathbb{E})$. If $\mu(B) = 0$, then $\mathbb{E}$ is finite-dimensional. In other words, only in finite-dimensional Banach spaces the unit ball is relatively compact.*

Proof: Suppose that $\overline{B}$ is compact, then there exists a finite set $X \subset \mathbb{E}$ such that $\overline{B} \subset X + \frac{1}{2}\overline{B}$. Put $\mathbb{E}_0 :=$ span (X). Clearly, dim $\mathbb{E}_0 < \infty$ and

$$\overline{B} \subset X + \frac{1}{2}\overline{B} \subset \mathbb{E}_0 + \frac{1}{2}\overline{B}. \tag{2.18}$$

By multiplying (2.18) by $\frac{1}{2}$, we get

$$\frac{1}{2}\overline{B} \subset \mathbb{E}_0 + \frac{1}{4}\overline{B}.$$

Thus,

$$\overline{B} \subset \mathbb{E}_0 + \mathbb{E}_0 + \frac{1}{4}\,\overline{B} = \mathbb{E}_0 + \frac{1}{4}\,\overline{B}.$$

By induction, for every $n \in \mathbb{N}$

$$\overline{B} \subset \mathbb{E}_0 + \frac{1}{2^n}\,\overline{B}, \text{ i.e. } \quad \overline{B} \subset \bigcap_{n-1}^{\infty}\left(\mathbb{E}_0 + \frac{1}{2^n}\,\overline{B}\right) = \mathbb{E}_0,$$

which implies

$$\mathbb{E} = \bigcup_{n=1}^{\infty} n\overline{B} \subset \bigcup_{n=1}^{\infty} n\mathbb{E}_0 = \mathbb{E}_0.$$

□

2.7.2 Compact, Completely Continuous, and Condensing Maps

Let μ be a measure of noncompactness on $\mathcal{B}(\mathbb{E})$. Then, μ can be extended to a measure of noncompactness on $\mathcal{B}(\mathbb{R}^n \oplus \mathbb{E})$ by

$$\mu(A) := \mu(\pi(A)), \quad A \in \mathcal{B}(\mathbb{R}^n \oplus \mathbb{E}),$$

where $\pi : \mathbb{R}^n \oplus \mathbb{E} \to \mathbb{E}$ is the natural projection.

Definition 2.7.5. Let μ be a measure of noncompactness on $\mathcal{B}(\mathbb{R}^n \oplus \mathbb{E})$. For $X \subset \mathbb{R}^n \oplus \mathbb{E}$, a continuous map $F : X \to \mathbb{E}$ is called

(i) a *μ-Lipschitzian* map with a constant $k \geq 0$, if $\mu(F(A)) \leq k\mu(A)$ for all $A \in \mathcal{B}(X)$;
(ii) a *compact* map, if X is bounded and $\mu(F(X)) = 0$;
(iii) a *completely continuous* map, if it is μ-Lipschitzian with a constant $k = 0$;
(iv) a *Darbó* map with constant $0 \leq k < 1$, if it is μ-Lipschitzian with the constant $k \in [0, 1)$;
(v) a *condensing* map, if it is μ-Lipschitzian with a constant $k = 1$ and $\mu(F(A)) < \mu(A)$ for every $A \in \mathcal{B}(X)$ such that $\mu(A) > 0$.

Definition 2.7.6. A bounded linear operator $L : \mathbb{R}^n \oplus \mathbb{E} \to \mathbb{E}$ is called *compact*, if L is a completely continuous map.

Proposition 2.7.7. *Let $G : \mathbb{R}^n \oplus \mathbb{E} \to \mathbb{E}$ be a Banach contraction with a constant $k \in [0, 1)$ and $K : \mathbb{R}^n \oplus \mathbb{E} \to \mathbb{E}$ a completely continuous map. Then $F(x) := G(x) + K(x)$ is a Darbó map with the same constant k, with respect to the Hausdorff measure χ of noncompactness defined by (2.16).*

Proof: For $y \in \mathbb{E}$, denote by $B_r^*(y)$ the ball of radius r centered at y in the target space $\mathbb{E}$. Take $A \in \mathcal{B}(\mathbb{R}^n \oplus \mathbb{E})$, and suppose $\alpha := \chi(A)$. Then, for every $\varepsilon > 0$, there exists a finite set $X = \{x_1, \dots, x_N\} \subset \mathbb{R}^n \oplus \mathbb{E}$ such that

$$A \subset \bigcup_{i=1}^{N} B_{\alpha+\varepsilon}(x_i).$$

Since G is a Banach contraction with a constant $k \in [0, 1)$, we obtain

$$G(A) \subset \bigcup_{i-1}^{N} G(B_{\alpha+\varepsilon}(x_i)) \subset \bigcup_{i-1}^{N} B^*_{k(\alpha+\varepsilon)}(G(x_i)) = G(X) + k(\alpha+\varepsilon)B_1^*(0).$$

Thus, $\chi(G(A)) \leq k(\alpha + \varepsilon)$, for any $\varepsilon > 0$, which implies that

$$\chi(G(A)) \leq k\chi(A).$$

On the other hand, by the properties of χ

$$\begin{aligned}\chi((G+K)(A)) &\leq \chi(G(A) + K(A)) \\ &\leq \chi(G(A)) + \chi(K(A)) = \chi(G(A)) \\ &\leq k\chi(A).\end{aligned}$$

Therefore, F is a Darbó map with the constant k. □

Proposition 2.7.8. *Let $U \subset \mathbb{R}^n \oplus \mathbb{E}$ be an open subset and $F : U \to \mathbb{E}$ a continuously Frechét differentiable Darbó map with a constant $k \in [0, 1)$. Then, for every $x_o \in U$, the derivative $L := DF(x_o) : \mathbb{R}^n \oplus \mathbb{E} \to \mathbb{E}$ is a Darbó operator with the same constant k.*

Proof: As before, we denote by $B_r(x)$ (resp. $B_r^*(y)$) the ball of radius r centered at $x \in \mathbb{R}^n \oplus \mathbb{E}$ (resp. at $y \in \mathbb{E}$). By the differentiability of F at $x_o \in U$, we have that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x - x_o\| < \delta$, then

$$\|F(x) - F(x_o) - L(x - x_o)\| \leq \varepsilon\|x - x_o\| < \varepsilon\delta,$$

which implies that

$$\|L(x - x_o)\| \leq \|F(x) - F(x_o)\| + \varepsilon\delta.$$

Then,

$$L(B_\delta(0) \subset F(B_\delta(x_o)) - F(x_o) + \varepsilon\delta B_1^*(0).$$

Therefore,

$$\begin{aligned}\delta\mu(L(B_1(0))) &= \mu(L(B_\delta(0)) \leq \mu(F(B_\delta(x_o) - F(x_o) + \varepsilon\delta B_1^*(0))\\ &\leq \mu(F(B_\delta(x_o)) + \varepsilon\delta\mu(B_1^*(0))\\ &\leq k\mu(B_\delta(x_o)) + \varepsilon\delta\mu(B_1^*(0))\\ &= k\delta\mu(B_1(0)) + \varepsilon\delta\mu(B_1^*(0)),\end{aligned}$$

which holds for every $\varepsilon > 0$, thus we have $\mu(L(B_1(0)) \leq k\mu(B_1(0))$. It follows that L is a Darbó operator with the constant k. □

Proposition 2.7.9. *(cf. [116]) Let* $L : \mathbb{E} \to \mathbb{E}$ *be a bounded Darbó operator. Then, the linear operator* $\mathrm{Id} - L : \mathbb{E} \to \mathbb{E}$ *is a bounded Fredholm operator of index zero.*

2.7.3 Completely Continuous and Condensing Fields

Definition 2.7.10. Let $\mathbb{E}$ be a Banach space, $X \subset \mathbb{R}^n \oplus \mathbb{E}$ and $f : X \to \mathbb{E}$ a continuous map of the form $f = \mathrm{Id} - F$, for $F : X \to \mathbb{E}$. Then, the map f is called

(i) a *compact field* on X, if F is a compact map;
(ii) a *completely continuous field* on X, if F is a completely continuous map;
(iii) a *Darbó field* on X, if F is a Darbó map;
(iv) a *condensing field* on X, if F is a condensing map.

A finite-dimensional degree theory can be extended in a standard way to the so-called *Leray-Schauder degree* theory for completely continuous fields on a Banach space $\mathbb{E}$. Further extensions of the degree theory can be done for Darbó fields and condensing fields on $\mathbb{E}$. For more details from this perspective, we refer to [116].

3

Primary Equivariant Degree: An Axiomatic Approach

The primary degree (with one parameter), as it was confirmed by a large number of possible applications (cf. [5, 6, 10, 13, 14, 17, 53, 55, 181, 118]), is one of the most effective tools for studying nonlinear equations with symmetries. In particular, it provides a unique alternative to the equivariant singularity method (cf. [79, 81, 94, 160]) for the treatment of symmetric Hopf bifurcation problems. However, the effectiveness of the primary degree is not just limited to symmetric bifurcation problems. This degree can also be applied to the existence problems (e.g. periodic solutions in autonomous system, see Chapter 8) based on the usage of the *a priori* bounds.

The primary degree (which was originally introduced in [72]) is a "part" of the general equivariant degree constructed by Ize *et al.* (cf. [97, 101]). The general equivariant degree is a full topological invariant (defined as an element of the stable equivariant homotopy group of sphere) expressing the obstruction for existence of an equivariant extension (without zeros) of a map from a boundary of a bounded region onto its interior. The primary degree turns out to be a computable part of the general equivariant degree. In this chapter, we present a new construction of the primary degree using normal approximations, fundamental domain techniques and connections to the classical Brouwer degree. In order to facilitate its applicability, we also provide for the primary degree a set of axioms (summarizing the main properties of the primary degree) and the computational result called the *recurrence formula* (cf. [114] for an earlier version in a slightly different setting), allowing its effective usage outside the equivariant topological context.

The *recurrence formula* reduces computations of the primary degree of an equivariant map to the computations of its S^1-degrees on the fixed-point subspaces. Since the S^1-equivariant degree plays a crucial role in a development of effective computational formulae for the primary degree, we derived a practical set of axioms for the S^1-degree and, based on these axioms established all the needed computational techniques. We also explore the notion of the so-called *basic maps* (i.e. the simplest equivariant maps having nontrivial primary G-degrees) with a particular attention given to basic S^1-maps. The obtained results allow further reductions of the computations, leading to a computerization* of the equivariant degree method.

The chapter is organized as follows. In Section 3.1, we recall the definition of the general equivariant degree and define the primary equivariant degree as its part. In Section 3.2, we present a new construction the primary equivariant degree via the usage of fundamental domains, where we indicate a direct connection of the primary degree with the (local) Brouwer degrees of related maps. The axiomatic definition of the primary degree is stated in Proposition 3.2.5. The notion of basic maps and $\mathbb{C}$-complementing maps are introduced in Section 3.3. Towards the computations of primary G-degree, we present a splitting lemma (cf. Lemma 3.3.4). Section 3.4 contains an axiomatic definition to the primary S^1-degree and several computational formulae as direct consequences of splitting lemma. In Section 3.5, we state and prove the recurrence formula in the context of the primary degree with n-parameters for $n \leq 1$.

3.1 General Equivariant Degree

Let G be a compact Lie group, V be an orthogonal G-representation, and $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded G-invariant subset. Consider a continuous Ω-admissible equivariant map $f : \overline{\Omega} \subset \mathbb{R}^n \oplus V \to V$, i.e. $f : (\overline{\Omega}, \partial\Omega) \to (V, V\setminus\{0\})$. One can assign to the pair (f, Ω) an element, called the *equivariant degree* and denoted by $\deg_G(f, \Omega)$, in the abelian group Π^G being *stable limit* of the equivariant homotopy groups Π_N of maps (cf. [72, 15])

$$S(\mathbb{R}^{N+n} \oplus V) \to S(\mathbb{R}^N \oplus V).$$

More precisely, take a large ball $B_R(\mathbb{R}^n \oplus V)$ such that $\overline{\Omega_\mathcal{N}} \subset B_R(\mathbb{R}^n \oplus V)$, where $\Omega_\mathcal{N} := \Omega \cup \mathcal{N}$ and $\mathcal{N}$ is an invariant neighborhood of $\partial\Omega$ such that $f(x) \neq 0$ for all $x \in \mathcal{N}$. Let $\eta : \overline{B_R(\mathbb{R}^n \oplus V)} \to \mathbb{R}$ be an invariant Urysohn function such that

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \notin \Omega_\mathcal{N}. \end{cases} \tag{3.1}$$

Define $F : \big([-1,1] \times \overline{B_R(\mathbb{R}^n \oplus V)}, \partial([-1,1] \times B_R(\mathbb{R}^n \oplus V))\big) \to (\mathbb{R} \oplus V, (\mathbb{R} \oplus V) \setminus \{0\})$ by

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[*] Special Maple © routines have been developed to assist effective computations of primary degree with one free parameter, for several interesting symmetry groups. The most recent version is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/~wkrawcew/degree`.

$$F(t,x) = (t + 2\eta(x), f(x)), \quad (t,x) \in [-1,1] \times \overline{B_R(\mathbb{R}^n \oplus V)}. \tag{3.2}$$

. The pair $\left([-1,1] \times \overline{B_R(0)}, \partial([-1,1] \times B_R(0))\right)$ is G-equivariantly homeomorphic to $\left(\overline{B(\mathbb{R} \oplus V)}, S(B(\mathbb{R} \oplus V))\right)$, so the map F determines an equivariant homotopy class $[F]$ in Π_1. Define

$$\deg_G(f,\Omega) := \xi_1[F] \in \Pi^G, \tag{3.3}$$

and call it a *G-equivariant degree* of f in Ω.

The equivariant degree constructed above, which is a slight modification of the construction given in [97], satisfies all the properties expected from any reasonable degree theory, like existence, homotopy invariance, excision, suspension, additivity etc (cf. [72, 15, 101]). Roughly speaking, the equivariant degree "measures" equivariant homotopy obstructions for $f_{|\partial\Omega}$ to have an equivariant extension without zeros over $\overline{\Omega}$.

As it is shown in [8], the group Π^G admits a splitting

$$\Pi^G = \bigoplus_{\dim W(H) \leq n} \Pi(H),$$

where $\Pi(H)$ consists of all the elements in Π^G generated by $B(\mathbb{R}^{N+n} \oplus V)$-admissible maps $f : \mathbb{R}^{N+n} \oplus V \to \mathbb{R}^N \oplus V$ being regular normal maps with zeros of the orbit type (H) only. Thus,

$$\deg_G(f,\Omega) = \sum_{\dim W(H) \leq n} n_{(H)}, \quad n_{(H)} \in \Pi(H).$$

If $\dim W(H) = n$, the component $\Pi(H)$ is called *primary*, and if $W(H)$ is bi-orientable, $\Pi(H) \simeq \mathbb{Z}$ (cf. [147]). The projection of $\deg_G(f,\Omega)$ onto $\sum\limits_{(H)\in\Phi_n^1(G,\Omega)} \Pi(H)$ is called the *primary degree of f in Ω* and is denoted by $G\text{-Deg}\,(f,\Omega)$.

The applicability of the primary degree depends heavily on its computability. In the general case $n > 0$, the computation of the primary degree is a complicated task. However, in the case $n = 1$, the primary degree seems to be completely computable due to a reduction to the S^1-degree using recurrence formula (cf. Sections 3.4—3.5). In the case $n = 2$, one can look for a similar reduction to the $S^1 \times S^1$-degree (cf. [97] for results on $S^1 \times S^1$-degree). In the case $n > 2$, the situation is much more complicated, since possible connected components of $W(H)$ may be different from n-tori.

3.2 Primary Equivariant Degree with n Free Parameters

The primary degree introduced in [72], uses the regular normal approximations and winding numbers of their restrictions to normal slices around the orbits of zeros (cf. [51, 52, 116], where the case $G = S^1$ was considered). Since it is well-known that the winding number admits an axiomatic definition as an integer-valued function satisfying a list of certain properties (cf. [112, 188]), it is natural to ask whether a similar axiomatic approach exists for the primary degree. The answer turns out to be affirmative.

3.2.1 Construction

Take an Ω-admissible G-equivariant map $f : \mathbb{R}^n \oplus V \to V$ and assume that it is regular normal in Ω. For $(H) \in \Phi_n^+(G, V)$, put $f_H := f_{|\Omega_H}$ and take a canonical orientation on $\Omega_H/W(H)$ (cf. Proposition 2.2.16). Choose a regular fundamental domain D on Ω_H such that $f_H^{-1}(0) \cap (D \setminus D_o) = \emptyset$ (cf. Section 2.5, Theorem 2.5.6). Put $T_o := p(D_o)$. Since f is regular normal, the set $p(f_H^{-1}(0) \cap D_o)$ is finite, thus it is always possible to construct T_o in such a way that $p(f_H^{-1}(0)) \subset T_o$. The homeomorphism $\xi := p^{-1}|_{T_o} : T_o \to D_o$ is called the *lifting homeomorphism*.

Definition 3.2.1. Consider an Ω-admissible G-equivariant regular normal map $f : \mathbb{R}^n \oplus V \to V$. We define the *primary degree* of f to be an element $G\text{-Deg}(f, \Omega) \in A_n^+(G)$ by

$$G\text{-Deg}(f, \Omega) := \sum_{i=1}^{r} n_{H_i}(H_i), \tag{3.4}$$

where the coefficient n_{H_i} corresponding to (H_i) is defined by

$$n_{H_i} := \deg(f_{H_i} \circ \xi, T_o), \tag{3.5}$$

with ξ being the lifting homeomorphism and deg standing for the (local) Brouwer degree of f_{H_i} (cf. Section 2.1.3). To certain extent, one can think of n_{H_i} as the Brouwer degree of f_{H_i} on a fundamental domain D.

If $f : \mathbb{R}^n \oplus V \to V$ is a general G-equivariant Ω-admissible map (not necessarily being regular normal in Ω), then take a regular normal approximation map $\tilde{f}$ of f (cf. Proposition 2.3.5) and define

$$G\text{-Deg}(f, \Omega) := G\text{-Deg}(\tilde{f}, \Omega). \tag{3.6}$$

We will show that the primary degree given by (3.4)—(3.6) is well-defined.

Proposition 3.2.2. *Let G be a compact Lie group, $\Omega \subset \mathbb{R}^n \oplus V$ an open bounded invariant subset and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. Then, the primary degree given by (3.4)—(3.6) is well-defined.*

Proof: (i) We first show that formula (3.5) is independent of a choice of a regular fundamental domain D. Suppose that D' is another regular fundamental domain such that $f_H^{-1}(0) \cap (D' \setminus D'_o) = \emptyset$, $p(D'_o) = T'_o$ with the lifting homeomorphism $\xi' : T'_o \to D'_o$. By applying the additivity property of the Brouwer degree, we can assume, without loss of generality, that $f_H^{-1}(0)$ is composed of a single orbit $W(H)(x_o)$ and put $p(x_o) = y_o$. Suppose that $B_o \subset T_o \cap T'_o$ is a contractible neighborhood of y_o, put $E_o = \xi(B_o)$, $E'_o = \xi'(B_o)$ and we assume $x_o \in E_o$. Then, by excision property of the degree,

$$\deg(f_H \circ \xi, T_o) = \deg(f_H \circ \xi, B_o), \quad \deg(f_H \circ \xi', T'_o) = \deg(f_H \circ \xi', B_o).$$

We will show that

$$\deg(f_H \circ \xi, B_o) = \deg(f_H \circ \xi', B_o). \tag{3.7}$$

Case 1. $x_o \in E_o \cap E'_o$. Observe that $\xi_{|B_o}$ and $\xi'_{|B_o}$ are sections of the (trivial) bundle $p : p^{-1}(B_o) \to B_o$, thus there exists a continuous map $\mu : E_o \to W(H)$ such that for every $x \in E_o$, we have

$$\Psi(x) := \mu(x)x \in E'_o$$

and $\Psi : E_o \to E'_o$ is a homeomorphism since so are $\xi_{|B_o}$ and $\xi'_{|B_o}$. In particular, $\mu(x_o) = 1$ and E_o is contractible. Therefore, there exists a homotopy μ_t of μ with a constant map $\mu_o(x) \equiv 1$. Put $\Psi_t(x) := \mu_t(x)x$, i.e. Ψ_t is a homotopy between Ψ and $\mathrm{Id}_{|E_o}$. Observe that $\xi' = \Psi \circ \xi$, therefore, by the homotopy invariance of the degree, we have

$$\deg(f_H \circ \xi', B_o) = \deg(f_H \circ \Psi \circ \xi, B_o) = \deg(f_H \circ \Psi_t \circ \xi, B_o) = \deg(f_H \circ \xi, B_o).$$

Case 2. $x_o \notin E_o \cap E'_o$. In this case, there exists $g \in W(H)_o$ such that $gx_o =: x'_o \in E'_o$. Put $\widetilde{D}_o := g(D_o)$. Since $W(H)_o$ acts freely, $\widetilde{D} := \widetilde{\widetilde{D}_o}$ is a fundamental domain with a lifting homeomorphism $\widetilde{\xi} = g \circ \xi$, and we put $\widetilde{E}_o = g(E_o)$. By the Sard-Brown theorem (cf. Proposition 2.1.5), we can assume that y_o is a regular point of the map $f_H \circ \xi$. Since f_H is $W(H)$-equivariant, we have

$$f_H \circ \xi = f_H \circ g^{-1} \circ g \circ \xi = g^{-1} \circ f_H \circ g \circ \xi = g^{-1} \circ f_H \circ \tilde{\xi},$$

i.e.

$$g \circ f_H \circ \xi = f_H \circ \tilde{\xi},$$

which implies that y_o is also a regular point of $f_H \circ \tilde{\xi}$. Since the action of $W(H)$ preserves the orientation of the slice, we obtain immediately

$$\deg(f_H \circ \xi, B_o) = \deg(f_H \circ \tilde{\xi}, B_o).$$

Since $x'_o \in E'_o \cap \widetilde{E}_o$, the equality (3.7) follows from the Case 1.

(ii) We show that the formula (3.5) does not depend on a choice of a representative f. Take two regular normal G-equivariant maps f_0 and $\tilde{f}_1$, which are equivariantly homotopic by an Ω-admissible homotopy $\Psi : [0,1] \times \mathbb{R}^n \oplus V \to V$ with $\Psi_0 = f_0$ and $\Psi_1 = \tilde{f}_1$ (where $\Psi_t := \Psi(t,\cdot)$). Let $(H) \in \Phi_n(G,V)$ and choose D^1 to be a regular fundamental domain for the $W(H)$-action on Ω_H such that $(f_0)_H^{-1}(0) \cap (D^1 \setminus D_o^1) = \emptyset$. Denote by $\xi^1 := (p_{|D_o^1})^{-1} : T_o^1 \to D_o^1$ the corresponding lifting homeomorphism. Then, by continuity of Ψ, there exists $0 < \tilde{t}_1 \leq 1$ such that $\bigcup_{t \in [0,\tilde{t}_1)} (\Psi_t)_H^{-1}(0) \cap (D^1 \setminus D_o^1) = \emptyset$. Since for every $t_1 \in [0, \tilde{t}_1)$, the map Ψ_t, $t \in [0, t_1]$, is a regular normal homotopy between f_0 and $f_1 := \Psi_{t_1}$, it follows from the homotopy property of the local Brouwer degree that

$$\deg((f_0)_H \circ \xi^1, T_o^1) = \deg((f_1)_H \circ \xi^1, T_o^1).$$

By the compactness of $[0,1]$, there exists a (finite) partition $0 < t_1 < \cdots < t_k = 1$ and fundamental domains $D^1, D^2, \ldots, D^k$ with the corresponding lifting homeomorphisms $\xi^i := (p_{|D_o^i})^{-1} : T_o^i \to D_o^i$, such that

$$\bigcup_{t \in [t_{i-1}, t_i]} (\Psi_t)_H^{-1}(0) \cap (D^i \setminus D_0^i) = \emptyset.$$

Consequently, by induction, we obtain

$$\deg((f_0)_H \circ \xi^1, T_o^1) = \deg((f_1)_H \circ \xi^1, T_o^1) = \cdots = \deg((f_k)_H \circ \xi^k, T_o^k),$$

which implies

$$\deg((f_0)_H \circ \xi^1, T_o^1) = \deg((f_k)_H \circ \xi^k, T_o^k).$$

Thus, Proposition 3.2.2 is proved.

□

We will proceed with the basic properties satisfied by the primary degree. To formulate the so-called normalization property, we start with the definition:

Definition 3.2.3. Let G be a compact Lie group, V an orthogonal G-representation and $f : \mathbb{R}^n \oplus V \to V$ a regular normal map such that $f(x_o) = 0$ with $G_{x_o} = H$ and $(H) \in \Phi_n^+(G, V)$.

(i) Let $U_{G(x_o)}$ be a G-invariant tubular neighborhood around $G(x_o)$ such that $f^{-1}(0) \cap U_{G(x_o)} = G(x_o)$. Then, f is called a *tubular map* around $G(x_o)$.
(ii) In addition, take a positively oriented slice S_{x_o} to $W(H)(x_o)$ in $\mathbb{R}^n \oplus V^H$ (cf. Definition 2.2.17). Call $n_{x_o} = \text{sign}\det Df^H(x_o)|_{S_{x_o}}$ the *local index* of f at x_o in $U_{G(x_o)}$ (here $f^H := f|_{\Omega^H}$ and D stands for the derivative).

Proposition 3.2.4. (cf. [72, 101]). *Let G, V, Ω and f be as in Proposition 3.2.2. Then the primary degree defined by (3.4)–(3.6) satisfies the following properties:*

(P1) (Existence) *If $G\text{-Deg}\,(f, \Omega) = \sum_{(H)} n_H(H)$ is such that $n_{H_o} \neq 0$ for some $(H_o) \in \Phi_n^+(G, V)$, then there exists $x \in \Omega$ with $f(x) = 0$ and $G_x \supset H_o$.*

(P2) (Additivity) *Assume that Ω_1 and Ω_2 are two G-invariant open disjoint subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$G\text{-Deg}\,(f, \Omega) = G\text{-Deg}\,(f, \Omega_1) + G\text{-Deg}\,(f, \Omega_2).$$

(P3) (Homotopy) *Suppose $h : [0,1] \times \mathbb{R}^n \oplus V \to V$ is an Ω-admissible G-equivariant homotopy. Then,*

$$G\text{-Deg}\,(h_t, \Omega) = \text{constant}$$

(here $h_t := h(t, \cdot, \cdot)$, $t \in [0, 1]$).

(P4) (Suspension) *Suppose that W is another orthogonal G-representation and let U be an open, bounded G-invariant neighborhood of 0 in W. Then,*

$$G\text{-Deg}\,(f \times \text{Id}, \Omega \times U) = G\text{-Deg}\,(f, \Omega).$$

(P5) (Normalization) *Suppose f is a tubular map around $G(x_o)$ with $H := G_{x_o}$ and $(H) \in \Phi_n^+(G, V)$. Let n_{x_o} be the local index of f at x_o in a tubular neighborhood $U_{G(x_o)}$. Then,*

$$G\text{-Deg}\,(f, U_{G(x_o)}) = n_{x_o}(H).$$

(P6) (Elimination) *Suppose f is normal in Ω and $\Omega_H \cap f^{-1}(0) = \emptyset$ for every $(H) \in \Phi_n^+(G,V)$. Then,*

$$G\text{-Deg}\,(f,\Omega) = 0.$$

(P7) (Excision) *If $f^{-1}(0) \cap \Omega \subset \Omega_0$, where $\Omega_0 \subset \Omega$ is an open invariant subset, then*

$$G\text{-Deg}\,(f,\Omega) = G\text{-Deg}\,(f,\Omega_0).$$

(P8) (Hopf property) *Suppose that $\Omega \subset \mathbb{R}^n \oplus V$ is an open invariant subset such that $\Omega_H/W(H)$ is connected for all $(H) \in \Phi_n^1(G,V)$ and $\Omega_K = \emptyset$ for all $(K) \in \Phi_k(G,V)$ with $k < n$ and all $(K) \in \Phi_n(G,V) \setminus \Phi_n^+(G)$. Let $f, g : \mathbb{R}^n \oplus V \to V$ be two Ω-admissible G-equivariant maps such that*

$$G\text{-Deg}\,(f,\Omega) = G\text{-Deg}\,(g,\Omega).$$

Then, f and g are G-equivariantly homotopic by an Ω-admissible homotopy.

Proof: (P1): Assume f is regular normal and $(H_o) \in \Phi_n^+(G)$. Choose a regular fundamental domain D and the lifting homeomorphism $\xi : T_o \to D_o$ for the $W(H_o)$-action on Ω_{H_o}. By assumption, $0 \neq n_{H_o} = \deg\,(f_{H_o} \circ \xi, T_o)$. Then, by the existence property of the (local) Brouwer degree, there exists $y_o \in T_o$ such that $f_{H_o}(\xi(y_o)) = 0$, i.e., $f_{H_o}(x_o) = 0$, where $x_o = \xi(y_o) \in D_o \subset \Omega_{H_o}$, so that $G_{x_o} = H_o$.

In the general case, take a sequence $\{f_n\}$ of G-equivariant Ω-admissible regular normal maps such that

$$\sup_{x\in\Omega} \|f_n(x) - f(x)\| < \frac{1}{n}.$$

Since for n sufficiently large f_n is G-equivariantly homotopic to f, it follows that $G\text{-Deg}\,(f,\Omega) = G\text{-Deg}\,(f_n,\Omega)$. Since f_n is normal, we obtain $f_n^{-1}(0) \cap \Omega_{H_o} \neq \emptyset$, thus there is a sequence $\{x_n\} \subset \Omega_{H_o}$ such that $f_n(x_n) = 0$ for each n sufficiently large. We can assume without loss of generality that $x_n \to x$ as $n \to \infty$ and therefore $f(x) = \lim_{n\to\infty} f_n(x_n) = 0$. Since V^{H_o} is closed, $x \in V^{H_o}$ and consequently $G_x \supset H_o$.

(P2) — (P4), (P7): To establish these properties, one can use the same idea as above: for a regular normal f (resp. h) the statements follow from (3.4), (3.5) and appropriate properties of the local Brouwer degree. In the general case, it suffices to take regular normal approximations sufficiently closed to f (resp. h) and use the standard compactness argument.

(P5): Follows from the regular value definition of the Brouwer degree.

(P6): Follows from the definition of the primary equivariant degree.

(P8): We divide the proof in several steps:

Step 1. Local homotopies around zeros: Denote by $\Phi_{n,0}(G,V)$ the set of all orbit types in $f^{-1}(0)\cap\Omega$, which is an invariant over the choice of all Ω-admissible G-equivariant maps taking the same value of G-Deg as f (by the definition of the G-Deg). In particular, it is also the set of all orbit types in $g^{-1}(0)\cap\Omega$. Without loss of generality (see Proposition 2.3.5), one can assume that f and g are regular normal. Further, by the assumption and regular normality (see Proposition 2.3.4), f and g only have zeros of primary orbit type. For each $(H)\in\Phi_{n,0}(G,V)$, choose a regular fundamental domain D on Ω_H provided by Theorem 2.5.6 with $T_o=p(D_o)$ such that $f_H^{-1}(0)\cap(D\setminus D_o)=\emptyset$ and $g_H^{-1}(0)\cap(D\setminus D_o)=\emptyset$, i.e. $p(f_H^{-1}(0))\cup p(g_H^{-1}(0))\subset T_o$. Notice that T_o is contractible (in particular, connected). Thus, by the Hopf Property of Brouwer degree,

$$\deg(f_H\circ\xi,T_o)=\deg(g_H\circ\xi,T_o)$$

implies that f_H is homotopic to g_H by a certain homotopy h_H on Ω_H. This homotopy can be extended, in a standard way (cf. [120, 47]), to a G-equivariant homotopy between f and g on $\Omega_{(H)}$. By Proposition 2.3.5, this homotopy can also be assumed to be regular and normal. Then, by using the normality condition, such a homotopy can be extended to an invariant neighborhood of $\Omega_{(H)}$, say $\mathcal{N}_{\Omega_{(H)}}$ (denote this homotopy by h_H). Apply the same argument to each $(H)\in\Phi_{n,0}(G,V)$ and choose for any (H) an invariant closed neighborhood $N_H\subset\mathcal{N}_{\Omega_{(H)}}$ satisfying the conditions: (i) N_H contains zeros of f and g of orbit type (H); (ii) $N_H\cap N_L=\emptyset$ as $(H)\neq(L)$. The collection of the "local" homotopies $\{h_{H_{|N_H}}\}$ for all $(H)\in\Phi_{n,0}(G,V)$, gives rise to the equivariant homotopy between f and g on the closed invariant subset $N:=\bigsqcup N_H$.

Step 2. Extension of local homotopies: based on the local homotopies, define a map h on $A:=(\{0\}\times\overline{\Omega})\cup([0,1]\times N)\cup(\{1\}\times\overline{\Omega})$ by letting $h(0,\cdot)=f(\cdot)$, $h(1,\cdot)=g(\cdot)$ and $h(t,x)=h_H(t,x)$ for $(t,x)\in[0,1]\times N$ and x of orbit type (H). By construction, h is continuous G-equivariant. Using the equivariant Kuratowski-Dugundji Theorem (see, for instance, [120], Theorem 1.3), extend h equivariantly and continuously over $[0,1]\times\overline{\Omega}$ and denote this extension by $\hat{h}$. In general, $\hat{h}$ may have new zeros.

Step 3. Correcting $\hat{h}$ via Urysohn function: Put $\hat{A} := \hat{h}^{-1}(0) \setminus A$ (i.e. the set of the "new zeros" of $\hat{h}$). We claim that $\hat{A}$ is a closed subset in $[0,1] \times \overline{\Omega}$. Indeed, take a sequence $\{(t_n, x_n)\}$ from $\hat{A}$, and suppose $\{(t_n, x_n)\} \to (t_o, x_o)$ in $[0,1] \times \overline{\Omega}$. By continuity of $\hat{h}$, we have $\hat{h}^{-1}(0)$ is a closed subset in $[0,1] \times \Omega$, so $(t_o, x_o) \in \hat{h}^{-1}(0)$. By the normality of h, one has: $(t_o, x_o) \notin A$, i.e. $\hat{A}$ is closed. By construction, $\hat{A} \cap A = \emptyset$, thus there exists an invariant Urysohn function $\eta : [0,1] \times \overline{\Omega} \to [0,1]$ with $\eta(A) = 1$ and $\eta(\hat{A}) = 0$. Now, define a new map $\tilde{h}$ on $[0,1] \times \Omega$ by: $\tilde{h}(t,x) = \hat{h}(t \cdot \eta(t,x), x)$. It's easy to see that $\tilde{h}^{-1}(0) = h^{-1}(0)$, thus $\tilde{h}$ is a required homotopy between f and g.

□

3.2.2 Axiomatic Definition

We are now in a position to state an axiomatic definition of the primary equivariant degree.

Proposition 3.2.5. *Let G be a compact Lie group. There exists a unique function* G-Deg *assigning to each admissible pair (f, Ω) an element* G-Deg$(f, \Omega) = \sum n_H(H)$ *in $A_n^+(G)$, which satisfies properties (P1)—(P6) listed in Proposition 3.2.4.*

Proof: The *existence* part of Proposition 3.2.5 is provided by Propositions 3.2.2 and 3.2.4. To prove the *uniqueness*, take an arbitrary admissible pair (f, Ω). By the homotopy property, f can be assumed to be regular normal. By additivity (i.e. excision) and elimination properties, we can assume that $\Omega \cap f^{-1}(0)$ contains points of the orbit types $(H) \in \Phi_n^+(G, V)$. Since f is regular normal, the set $\Omega \cap f^{-1}(0)$ is composed of a finite number of G-orbits. Take tubular neighborhoods isolating the above orbits (this is durable, since we have finitely many zero orbits). By the additivity, the primary degree of (f, Ω) is equal to the sum of degrees of restrictions of f to the tubular neighborhoods. By the elimination axiom, the contribution of the secondary orbit types is equal to zero. Finally, by the normalization property, the remaining orbits lead to "local indices", which determine uniquely the value of the primary degree G-Deg(f, Ω). □

We provide a computational example for the primary degree with 2-parameters.

Example 3.2.6. Let ${}^m\mathcal{V}$ and ${}^l\mathcal{V}$ be the m-th and l-th irreducible representation of S^1 (cf. Appendix A2) with $m, l > 0$. Put $V := {}^m\mathcal{V} \oplus {}^l\mathcal{V}$, which is naturally a T^2-representation. Define a map $d : \mathbb{R}^2 \oplus V \to V$ by

$$d(s,t,z,w) := \big((1 - \|z\| + i(s+t)) \cdot z, (1 - \|w\| + i(s-t)) \cdot w\big),$$

for $s, t \in \mathbb{R}$, $z \in {}^m\mathcal{V}$ and $w \in {}^l\mathcal{V}$. Let $\Omega \subset \mathbb{R}^2 \oplus V$ be defined by

$$\Omega := \{(s,t,z,w) \in \mathbb{R}^2 \oplus V \ : \ s^2 + t^2 < 1, \ \frac{1}{2} < \|z\|, \|w\| < 2\}.$$

Clearly, the map d is a T^2-equivariant Ω-admissible map. Also, by direct verification, the zero set $d^{-1}(0) \cap \Omega$ is composed of only one T^2-orbit $\{(s,t,z,w) \ : \ s = t = 0, \|z\| = \|w\| = 1\}$, which is of orbit type $(\mathbb{Z}_m \times \mathbb{Z}_l)$. Moreover, the map d is a regular normal map on Ω, since $\Omega = \Omega_{(\mathbb{Z}_m \times \mathbb{Z}_l)}$. Therefore, by normalization property, we have

$$T^2\text{-Deg}\,(d, \Omega) = i \cdot (\mathbb{Z}_m \times \mathbb{Z}_l),$$

where i is the local index of d at some point x_o in the orbit. For simplicity, choose $x_o := (0,0,1,0,1,0)$ written in real coordinates. Then, the slice $S_{x_o} \simeq \{(s,t,x_1,y_1,x_2,y_2) \ : \ y_1 = y_2 = 0\}$. Calculating the derivative $Dd(x_o)$ on the slice S_{x_o}, we have

$$Dd(x_o)|_{S_{x_o}} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

and $\det\big(Dd(x_o)|_{S_{x_o}}\big) > 0$. Notice that S_{x_o} is a negatively oriented slice (cf. Definition 2.2.17). Therefore, $T^2\text{-Deg}\,(d, \Omega) = -(\mathbb{Z}_m \times \mathbb{Z}_l)$.

3.3 Basic Maps, ℂ-Complementing Maps and Splitting Lemma

3.3.1 Basic Maps and ℂ-Complementing Maps

The S^1-degree will be the main computational tool to evaluate the primary G-degree for $G = \Gamma \times S^1$, with Γ being a compact Lie group. In order to establish the links between the S^1-degree and the primary G-degree, we introduce the notion of the so-called *basic maps* and *ℂ-complementing maps*. These two types of equivariant maps (which can be considered as the simplest nontrivial examples of G-equivariant maps with one parameter) appear naturally in the setting related to the existence of periodic solution problems (basic maps) and symmetric Hopf bifurcation problems ($\mathbb{C}$-complementing maps). The important feature of these types of maps is that they have *exactly* the same primary G-degrees. In fact, the $\mathbb{C}$-complementing map can be viewed as a "suspension" of the basic map (cf. Proposition 3.3.1).

Let $G = \Gamma \times S^1$ and V be an orthogonal G-representation with $V^{S^1} = \{0\}$ (notice that the S^1-action induces a natural complex structure on V). Put

$$\Omega := \{(t, v) \in \mathbb{R} \oplus V \ : \ |t| < 1, \ \frac{1}{2} < \|v\| < 2\}, \tag{3.8}$$

$$\mathcal{O} := \left\{(\lambda, v) \in \mathbb{C} \oplus V : \|v\| < 2, \ \frac{1}{2} < |\lambda| < 4\right\}. \tag{3.9}$$

Suppose $a : S^1 \to GL^G(V)$ is a continuous map. We define $b : \mathbb{R} \oplus V \to V$ and $f_a : (\mathbb{C} \setminus \{0\}) \times V \to \mathbb{R} \oplus V$ by

$$b(t, v) := \big(1 - \|v\| + it\big) \cdot v, \quad t \in \mathbb{R}, \ v \in V, \tag{3.10}$$

$$f_a(\lambda, v) := \left(|\lambda|(\|v\| - 1) + \|v\| + 1, a\left(\frac{\lambda}{|\lambda|}\right) v\right). \tag{3.11}$$

Similarly, define

$$f_a^-(\lambda, v) := \left(|\lambda|(\|v\| - 1) + \|v\| + 1, a\left(\frac{\bar{\lambda}}{|\lambda|}\right) v\right), \tag{3.12}$$

$$b^-(t, v) := \big(1 - \|v\| - it\big) \cdot v, \quad t \in \mathbb{R}, \ \ v \in V. \tag{3.13}$$

It is easy to check that the pairs (b, Ω), (b^-, Ω), $(f_a, \mathcal{O})$, $(f_a^-, \mathcal{O})$ are admissible pairs.

Proposition 3.3.1. *Let $G = \Gamma \times S^1$ for Γ being a compact Lie group, V be an orthogonal G-representation such that $V^{S^1} = \{0\}$ and Ω, $\mathcal{O}$ are given by (3.20) and (3.9). Assume that $a(\lambda) = \frac{\lambda}{|\lambda|}\mathrm{Id} : V \to V$, $\lambda \in \mathbb{C} \setminus \{0\}$ and consider the maps b and f_a defined by (3.19) and (3.11). Then, f_a is G-homotopic to a map f_1, which is a suspension of b on an open subset of zeros of f_1. In particular,*

$$G\text{-Deg}\,(f, \mathcal{O}) = G\text{-Deg}\,(b, \Omega). \tag{3.14}$$

Moreover, a similar property holds for b^- and f_a^- (defined by (3.12) and (3.22))

$$G\text{-Deg}\,(f^-, \mathcal{O}) = G\text{-Deg}\,(b^-, \Omega). \tag{3.15}$$

Proof: We only prove (3.26) and the proof of (3.27) is similar. Consider the map

$$f_a(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, \frac{\lambda}{|\lambda|} \cdot v \right).$$

Define the function $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2} \\ t - \frac{1}{2} & \text{if } \frac{1}{2} \le t \le \frac{3}{2} \\ 1 & \text{if } t > \frac{3}{2}, \end{cases}$$

and $\theta(v) := \eta(\|v\|)$, for $v \in V$. Put

$$f_\theta(\lambda, v) = (1 - \theta(v))\big(f_1(\lambda, 0) + v\big) + \theta(v) f_1(\lambda, v), \tag{3.16}$$

where $(\lambda, v) \in \overline{\mathcal{O}}$. Obviously, f_1 is G-homotopic to f_θ by an $\mathcal{O}$-admissible homotopy, so we have

$$G\text{-Deg}\,(f_1, \mathcal{O}) = G\text{-Deg}\,(f_\theta, \mathcal{O}), \quad \theta \in [0, 1].$$

By direct verification, $f_\theta^{-1}(0) = Z_0 \cup Z_1 \subset \Omega$, where $Z_0 := \Big\{ (\lambda, 0) \in \mathbb{C} \oplus V : |\lambda| = 1 \Big\}$ and $Z_1 := \Big\{ (-3, v) \in \mathbb{C} \oplus V : \|v\| = 1 \Big\}$. Define G-invariant open tubular neighborhoods Ω_0 and Ω_1 around Z_0 and Z_1 respectively by

$$\Omega_0 := \left\{ (\lambda, v) : \frac{1}{2} < |\lambda| < \frac{3}{2},\ \|v\| < \frac{1}{2} \right\}$$

and

$$\Omega_1 := \left\{ (\lambda, v) : |\lambda + 3| < \frac{1}{2},\ \frac{1}{2} < \|v\| < \frac{3}{2} \right\}.$$

By the additivity property, we have

$$G\text{-Deg}\,(f_\theta, \mathcal{O}) = G\text{-Deg}\,(f_\theta, \Omega_0) + G\text{-Deg}\,(f_\theta, \Omega_1).$$

Since for $(\lambda, v) \in \Omega_0$, we have $f_\theta(\lambda, v) = (1 - |\lambda|, v)$, it follows from the suspension property that

$$G\text{-Deg}\,(f_\theta, \Omega_0) = G\text{-Deg}\,(\varphi_o, B_o),$$

where $B_o = \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < 3\}$ and $\varphi_o : \overline{B}_o \to \mathbb{R}$ is defined by $\varphi_o(\lambda) = 1 - |\lambda|$. By the elimination property, we have $G\text{-Deg}\,(\varphi_o, B_o) = 0$. Thus,,

$$G\text{-Deg}\,(f, \mathcal{O}) = G\text{-Deg}\,(f_o, \Omega_1).$$

Replacing in the $\mathbb{R}$-component of (3.16) $\theta(v)$ (resp. $\|v\|$) by $\|v\| - \frac{1}{2}$ (resp. 1), one obtains the map

$$\begin{aligned}\tilde{f}_o(\lambda, v) &= \left(\frac{1}{2}(3 - |\lambda|), \left(1 - \theta(v) + \theta(v) \cdot \frac{\lambda}{|\lambda|}\right) \cdot v\right)\\ &= \left(\frac{1}{2}(3 - |\lambda|), \frac{3(1 + |\lambda|) - (2|\lambda| + 6)\|v\| + (2\|v\| - 1)(\lambda + 3)}{2|\lambda|} \cdot v\right)\end{aligned}$$

where $(\lambda, v) \in \Omega_1$ (recall, $\theta(v) = \|v\| - \frac{1}{2}$ on Ω_1).

Obviously, $\tilde{f}_o$ has no zeros on $\partial\Omega_1$. Moreover, for any $(\lambda, v) \in \partial\Omega_1$ the vectors $f_o(\lambda, v)$ and $\tilde{f}_o(\lambda, v)$ do not point the opposite directions. Therefore, f_o and $\tilde{f}_o$ are G-homotopic by Ω_1-admissible homotopy and

$$G\text{-Deg}\,(f_o, \Omega_1) = G\text{-Deg}\,(\tilde{f}_o, \Omega_1).$$

Next, replacing in the V-component of $\tilde{f}_o$ the value $|\lambda|$ (resp. $2\|v\| - 1$) by 3 (resp. 1), one obtains the map

$$\tilde{f}_1(\lambda, v) = \left(\frac{1}{2}(3 - |\lambda|), \frac{12(1 - \|v\|) + (\lambda + 3)}{6} \cdot v\right),$$

where $(\lambda, v) \in \Omega_1$.

At this moment, we can apply the change of variables $\lambda' = \lambda + 3$, leading to the set $\Omega_2 := \left\{(\lambda', v) : |\lambda'| < \frac{1}{2},\ \frac{1}{2} < \|v\| < \frac{3}{2}\right\}$ and after an appropriate S^1-homotopy) the map $\tilde{f}_1 : \overline{\Omega_2} \to \mathbb{R} \oplus V$, given by

$$\tilde{f}_1(\alpha + i\beta, v) = \left(\frac{1}{2}\alpha, \frac{12(1 - \|v\|) + (\alpha + i\beta)}{6} \cdot v\right), \quad \lambda' = \alpha + i\beta,$$

(here, we used the fact that $3 - |\lambda| = 3 - \sqrt{(\alpha - 3)^2 + (\beta)^2}$ is S^1-homotopic to α, since $|\beta| \leq |\lambda'| < \frac{1}{2}$, which guarantees no zeros of such a homotopy crossing $\partial\Omega_2$), which is clearly Ω_2-admissibly S^1-homotopic to the map

$$\overline{f_1}(\alpha + i\beta, v) = (\alpha, (1 - \|v\| + i\beta) \cdot v).$$

Obviously, $\overline{f}_1$ is a suspension of the map b, therefore

$$G\text{-Deg}\,(\overline{f_1}, \Omega_2) = G\text{-Deg}\,(b, \Omega),$$

and since

$$G\text{-Deg}\,(\overline{f_1}, \Omega_2) = G\text{-Deg}\,(\tilde{f}_0, \Omega_1) = G\text{-Deg}\,(f, \mathcal{O}),$$

the equality (3.26) follows. □

Definition 3.3.2. Let $\mathcal{V}_{j,l}$ be an irreducible representation of $\Gamma \times S^1$ (cf. Remark 2.2.4), and $\Omega_{j,l}$, $\mathcal{O}_{j,l}$ be defined by (3.20) and (3.9) with V replace by $\mathcal{V}_{j,l}$. We call the map b defined by (3.19) the *basic map* associated with $\mathcal{V}_{j,l}$, and the map b^- defined by (3.22) is called the *basic map of second type*. The map f_a defined by (3.11) is called the ℂ-*complementing map* and f_a^- defined by (3.12) will be called the ℂ-*complementing map of second type*. In addition, $(f, \mathcal{O}_{j,l})$ (resp. $(f^-, \mathcal{O}_{j,l})$) is called a ℂ-*complementing pair* to $(b, \Omega_{j,l})$ (resp. $(b^-, \Omega_{j,l})$).

By Proposition 3.3.1, we have the following

Corollary 3.3.3. *Let $G = \Gamma \times S^1$ for Γ being a compact Lie group and $V_{j,l}$ be an irreducible G-representation. Suppose $\Omega_{j,l}$, $\mathcal{O}_{j,l}$, b, b^-, f_a and f_a^- are as given in Definition 3.3.2. Then, we have*

$$G\text{-Deg}\,(f, \mathcal{O}_{j,l}) = G\text{-Deg}\,(b, \Omega_{j,l}),$$
$$G\text{-Deg}\,(f^-, \mathcal{O}_{j,l}) = G\text{-Deg}\,(b^-, \Omega_{j,l}).$$

3.3.2 Splitting Lemma

Lemma 3.3.4. (Splitting Lemma) *Let $G = \Gamma \times S^1$ for a compact Lie group Γ, V_1 and V_2 orthogonal G-representations with $V_i^{S^1} = \{0\}$, $i = 1, 2$. Put $V := V_1 \oplus V_2$. Suppose that $a_i : S^1 \to GL^G(V_j)$, $i = 1, 2$, are two continuous maps and $a : S^1 \to GL^G(V)$ is given by*

$$a(\lambda) = a_1(\lambda) \oplus a_2(\lambda), \quad \lambda \in S^1.$$

Assume $\mathcal{O}$ and f_a are defined by (3.9) and (3.11), respectively. Put

$$\mathcal{O}_i := \left\{(\lambda, v_i) \in \mathbb{C} \oplus V_i : \|v_i\| < 2,\ \frac{1}{2} < |\lambda| < 4\right\},$$
$$f_{a_i}(\lambda, v_i) := \left(|\lambda|(\|v_i\| - 1) + \|v_i\| + 1, a_i\left(\frac{\lambda}{|\lambda|}\right) v_i\right),$$

where $i = 1, 2$, $v_i \in V_i$. Then,

$$G\text{-Deg}\,(f_a, \mathcal{O}) = G\text{-Deg}\,(f_{a_1}, \mathcal{O}_1) + G\text{-Deg}\,(f_{a_2}, \mathcal{O}_2).$$

Proof: We can assume without loss of generality that $a_i : S^1 \to GL^G(V_i) \cap O(V_i)$ is analytic, i.e. there exists an analytic extension of a_i to a neighborhood of S^1 in $\mathbb{C}$ (here $O(V_i)$ stands for the group of orthogonal operators on V_i, $i = 1, 2$). Introduce the functions $q_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$,

$$q_i(t) = \begin{cases} 1 & \text{if } 0 \le t < s_i; \\ -\frac{1}{\epsilon_i}(t - t_i) & \text{if } s_i \le t < t_i; \\ 0 & \text{if } t \ge t_i, \end{cases} \quad \text{where} \quad \begin{cases} s_i = \frac{i}{i+4} - \frac{1}{2(i+4)^2}; \\ t_i = \frac{i}{i+4} + \frac{1}{2(i+4)^2}; \\ \epsilon_i = t_i - s_i = \frac{1}{(i+4)^2}. \end{cases}$$

Then define for $(\lambda, v_1, v_2) \in \overline{\mathcal{O}} \subset \mathbb{C} \oplus V_1 \oplus V_2$ the map

$$\tilde{f}_a(\lambda, v_1, v_2) := \Big(\theta(\lambda, v_1, v_2), \beta_1(\lambda, v_1), \beta_2(\lambda, v_1, v_2)\Big),$$

with

$$\theta(\lambda, v_1, v_2) = |\lambda|(\|v_1 + v_2\| - 1) + \|v_1 + v_2\| + 1,$$

$$\beta_1(\lambda, v_1) = q_2(\|v_1\|)v_1 + (1 - q_2(\|v_1\|)a_1\left(\frac{\lambda}{|\lambda|}\right)v_1,$$

$$\beta_2(\lambda, v_1, v_2) = q_1(\|v_1 + v_2\|)v_2 + (1 - q_1(\|v_1 + v_2\|)a_2\left(\frac{\lambda}{|\lambda|}\right)v_2.$$

The maps f_a and $\tilde{f}_a$ are G-homotopic by an $\mathcal{O}$-admissible homotopy.

Let us examine zeros of the map $\tilde{f}_a$. It is clear that

$$Z_0 := \Big\{(\lambda, 0, 0) : |\lambda| = 1\Big\} \subset \tilde{f}_a^{-1}(0).$$

Observe that if $(\lambda, v_1, v_2) \in \tilde{f}_a^{-1}(0)$ is such that $v_1 \neq 0$ (resp. $v_2 \neq 0$) then $v_2 = 0$ (resp. $v_1 = 0$). Indeed, suppose that $(\lambda, v_1, v_2) \in \tilde{f}_a^{-1}(0)$ is such that $v_1 \neq 0 \neq v_2$. Then, by comparing the norms of the both sides in the following equalities: $q_2(\|v_1\|)v_1 = -(1 - q_2(\|v_1\|)a_1\left(\frac{\lambda}{|\lambda|}\right)v_1$ and $q_1(\|v_1 + v_2\|)v_2 = -(1 - q_1(\|v_1 + v_2\|)a_2\left(\frac{\lambda}{|\lambda|}\right)v_2$, we obtain

$$q_2(\|v_1\|) = 1 - q_2(\|v_1\|) \quad \text{and} \quad q_1(\|v_1 + v_2\|) = 1 - q_1(\|v_1 + v_2\|),$$

which implies

$$q_2(\|v_1\|) = q_1(\|v_1 + v_2\|) = \frac{1}{2},$$

so $\|v_1\| = \frac{1}{3}$ and $\|v_1 + v_2\| = \frac{1}{5}$, but this is a contradiction because v_1 is orthogonal to v_2 and thus $\|v_1 + v_2\| \geq \|v_1\|$.

Therefore, we can first suppose that $(\lambda, v_1, 0) \in \tilde{f}_a^{-1}(0)$, $v_1 \neq 0$, so $\|v_1\| = \frac{1}{3}$. Then $\theta(\lambda, v_1, 0) = 0$ and $\beta_1(\lambda, v_1) = 0$ imply $|\lambda|\left(\frac{1}{3} - 1\right) + \frac{1}{3} + 1 = 0$, i.e. $|\lambda| = 2$. On the other hand, since $q_2(\frac{1}{3}) = \frac{1}{2}$,

$$\beta_1(\lambda, v_1) = \frac{1}{2}\left[v_1 + a_1\left(\frac{\lambda}{|\lambda|}\right)v_1\right] = 0, \quad v_1 \neq 0,$$

λ satisfies the equation

$$\det_{\mathbb{C}}\left[\mathrm{Id} + a_1\left(\frac{\lambda}{|\lambda|}\right)\mathrm{Id}\right] = 0, \quad |\lambda| = 2. \tag{3.17}$$

Since the map $\omega \to \det_{\mathbb{C}}[\mathrm{Id} + a_1(\omega)\mathrm{Id}]$ is analytic in a neighborhood of S^1 in $\mathbb{C}$, the equation

$$\det_{\mathbb{C}}[\mathrm{Id} + a_1(\omega)\mathrm{Id}] = 0, \quad \omega \in S^1,$$

has only a finite number of solutions, and consequently the equation (3.17) also has finitely many solutions, say $\lambda_1, \ldots, \lambda_n$. Put

$$Z_k := \left\{(\lambda_k, v_1, 0) : \|v_1\| = \frac{1}{3}\right\}, \quad k = 1, \ldots, n.$$

If $(\lambda, v_1, 0) \in \tilde{f}_a^{-1}(0)$, $v_1 \neq 0$, then $(\lambda, v_1, 0) \in Z_1 \cup \cdots \cup Z_n$. Similarly, if $(\lambda, 0, v_2) \in \tilde{f}_a^{-1}(0)$, $v_2 \neq 0$, then $\|v_2\| = \frac{1}{5}$ and $|\lambda| = \frac{3}{2}$, and there exists a finite number of solutions $\lambda'_1, \ldots, \lambda'_m$ to the equation

$$\det_{\mathbb{C}}\left[\mathrm{Id} + a_2\left(\frac{\lambda}{|\lambda|}\right)\mathrm{Id}\right] = 0, \quad |\lambda| = \frac{3}{2}.$$

Put $Z'_l := \left\{(\lambda'_l, 0, v_2) : \|v_2\| = \frac{1}{5}\right\}$, $l = 1, \ldots, m$. In this way, we have proved that $\tilde{f}_a^{-1}(0) \subset Z_0 \cup Z_1 \cup \cdots \cup Z_n \cup Z'_1 \cup \cdots \cup Z'_m$. By applying the excision property to G-invariant separating neighborhoods of Z_k, Z'_l, $k = 0, 1, \ldots, n$, $l = 1, \ldots, m$, and using appropriate deformations of $\tilde{f}_a$ on these sets, we obtain the map $\hat{f}_a$ such that $\hat{f}_a(\lambda, v_1, v_2) = (\theta(\lambda, v_1, v_2), \beta_1(\lambda, v_1), v_2)$ for (λ, v_1, v_2) in a neighborhood of Z_k, $k = 1, \ldots, n$, and $\hat{f}_a(\lambda, v_1, v_2) = (\theta(\lambda, v_1, v_2), v_1, \beta_2(\lambda, 0, v_2))$ for (λ, v_1, v_2) in a neighborhood of Z'_l, $l = 1, \ldots, m$. Notice that $\hat{f}_a$ in a neighborhood of Z_0 is homotopic to a map without zeros.

The conclusion then follows from the suspension and excision properties. ☐

3.4 Primary Equivariant S^1-degree

To assist an effective computation of the primary equivariant degree with one free parameter, we formulate an axiomatic definition (of more practical meaning than the one provided in Proposition 3.2.5) for the S^1-degree, based on the usage of the basic maps and folding homomorphisms (cf. Definition 3.4.1). For the rest of this chapter, we assume that $n = 1$.

Denote by $A_1(S^1) := A_1^1(S^1)$ the free $\mathbb{Z}$-module generated by the symbols $(\mathbb{Z}_k)$, $k = 1, 2, 3, \ldots$.

Definition 3.4.1. Consider an orthogonal S^1-representation V, an open S^1-invariant bounded set $\Omega \subset \mathbb{R} \oplus V$, and an Ω-admissible S^1-equivariant map $f : \mathbb{R} \oplus V \to V$. The primary degree $S^1\text{-Deg}(f, \Omega)$, also called the *$S^1$-equivariant degree*, is an element in $A_1(S^1)$ and can be written as (cf. (3.4)–(3.6))

$$S^1\text{-Deg}(f, \Omega) = \sum_{i=1}^{r} n_{k_i}(\mathbb{Z}_{k_i}), \quad n_{k_i} \in \mathbb{Z}. \tag{3.18}$$

Notation 3.4.2 Denote by ${}^l\mathcal{V}$, for $l = 1, 2, 3, \ldots$, the l-th real irreducible representation of S^1 (cf. Appendix A2, Table A2.1). For each l, there is an associated basic map b (cf. Definition 3.3.2). To be more precise, $b : \mathbb{R} \oplus {}^l\mathcal{V} \to {}^l\mathcal{V}$ by

$$b(t, z) := \big(1 - |z| + it\big) \cdot z, \quad (t, z) \in \mathbb{R} \oplus {}^l\mathcal{V}. \tag{3.19}$$

We will also use the notation ${}^l\Omega$ for the admissible domain of b, i.e.

$${}^l\Omega := \left\{(t, z) \in \mathbb{R} \oplus {}^l\mathcal{V} : |t| < 1,\ \frac{1}{2} < |z| < 2\right\}. \tag{3.20}$$

To formulate an axiomatic definition of S^1-degree, we need the following:

Definition 3.4.3. For every integer $m = 1, 2, 3, \ldots$, we define the homomorphism $\theta_m : S^1 \to S^1$ by $\theta_m(\gamma) = \gamma^m$, $\gamma \in S^1$. Define the induced homomorphism $\Theta_m : A_1(S^1) \to A_1(S^1)$ by

$$\Theta_m(\mathbb{Z}_k) := (\mathbb{Z}_{km}), \quad k = 1, 2, 3, \ldots,$$

where $(\mathbb{Z}_k)$ are the free generators of $A_1(S^1)$, and call it the *m-folding homomorphism.*

Notice that if $f : \mathbb{R} \oplus V \to V$ is an Ω-admissible S^1-equivariant map for an open bounded S^1-invariant subset $\Omega \subset \mathbb{R} \oplus V$, then for every integer $m = 1, 2, 3, \dots$, we can define the *associated m-folded* S^1-representation ${}^m(V)$, which is the same vector space V with the S^1-action '$\cdot$' given by

$$\gamma \cdot v := \theta_m(\gamma)v = \gamma^m v, \quad \gamma \in S^1, \quad v \in V.$$

The map $f : \mathbb{R} \oplus {}^m(V) \to^m (V)$ is S^1-equivariant. The set Ω considered as an S^1-subset of $\mathbb{R} \oplus {}^m(V)$ will be denoted by ${}^m\Omega$. We will say that the pair $(f, {}^m\Omega)$ is the *m-folded* admissible pair *associated* with (f, Ω).

3.4.1 Axiomatic Definition

The following theorem provides us with an axiomatic definition of the S^1-degree.

Theorem 3.4.4. *There exists a unique function, denoted by* S^1-Deg, *assigning to each admissible pair* (f, Ω) *an element* $S^1\text{-Deg}\,(f, \Omega) \in A_1(S^1)$ *satisfying the properties (P1)–(P4) (see Proposition 3.2.4 with* $G = S^1$*) as well as the following ones:*

(P5)' (Normalization) *Let* ${}^1\mathcal{V}$ *be the first irreducible* S^1*-representation and* $b : \mathbb{R} \oplus {}^1\mathcal{V} \to {}^1\mathcal{V}$ *be the basic map associated with* ${}^1\mathcal{V}$ *(cf. Notation 3.4.2). Then, we have*

$$S^1\text{-Deg}\,(b, {}^1\Omega) = (\mathbb{Z}_1).$$

(P6)' (Elimination) *If* V *is a trivial* S^1*-representation, then*

$$S^1\text{-Deg}\,(f, \Omega) = 0.$$

(F) (Folding) *Let* ${}^m(V)$ *be the m-folded representation associated with* V*, and* $(f, {}^m\Omega)$ *the m-folded admissible pair associated with* (f, Ω)*. Then*

$$S^1\text{-Deg}\,(f, {}^m\Omega) = \Theta_m\big[S^1\text{-Deg}\,(f, \Omega)\big].$$

The proof of Theorem 3.4.4 is essentially based on the following lemma:

Lemma 3.4.5. *Let* $f : \mathbb{R} \oplus V \to V$ *be a regular normal* Ω*-admissible map such that* $f^{-1}(0) \cap \Omega$ *consists of one* S^1*-orbit* $G(x_o)$*. Suppose that* $G_{x_o} = \mathbb{Z}_{k_o}$ *and denote by* S_{x_o} *the positively oriented slice at* x_o *to the orbit* $G(x_o)$ *(cf. Definition 2.2.17). Then,*

$$S^1\text{-Deg}\,(f, \Omega) = n_o\,(\mathbb{Z}_{k_o}),$$

where n_o *is the local index of* f *at* x_o *(cf. Definition 3.2.3).*

Proof: *Step 1: Unfolding the S^1-Action.*

Consider the S^1-isotypical decomposition (2.6) of the space V. Assume that $x_o = y_0 + y_1 + \cdots + y_r$, where $y_i \in V_i$. Notice that $G_{x_o} = \bigcap_{y_i \neq 0}^{r} G_{y_i}$, where $G_{y_0} = S^1$ and $G_{y_i} = \mathbb{Z}_{k_i}$ for $1 \leq i \leq r$ and $y_i \neq 0$. Thus, we have $\mathbb{Z}_{k_o} = S^1 \cap \bigcap_{y_i \neq 0} \mathbb{Z}_{k_i}$, which implies that k_i is a multiple of k_o if $y_i \neq 0$.

In the case k_j is not a multiple of k_o, the isotypical component V_{k_j} is orthogonal to $\mathbb{R} \oplus V^{\mathbb{Z}_{k_o}}$. By the normality assumption of f, on a small neighborhood of $G(x_o)$, f can be considered as the product map $f_o \times \text{Id}$, with $f_o := f_{|\mathbb{R} \oplus V^{\mathbb{Z}_{k_o}}}$. By the suspension property (P4), we have

$$S^1\text{-Deg}\,(f, \Omega) = S^1\text{-Deg}\,(f_o \times \text{Id}, \Omega_o \times B) = S^1\text{-Deg}\,(f_o, \Omega_o),$$

where $\Omega_o = \Omega \cap (\mathbb{R} \oplus V^{\mathbb{Z}_{k_o}})$ and B denotes the unit ball in $(\mathbb{R} \oplus V^{\mathbb{Z}_{k_o}})^{\perp}$. Thus,

$$\text{sign det}\, Df(x_o)|_{S_{x_o}} = \text{sign det}\, Df_o(x_o)|_{S'_{x_o}},$$

where $S'_{x_o} := S_{x_o} \cap (\mathbb{R} \oplus V^{\mathbb{Z}_{k_o}})$.

Thus, we can assume, without loss of generality, that $k_i = k_o \cdot n_i$ for $n_i \in \mathbb{Z}$ and $k_o = gcd(k_1, \dots, k_r)$. In this case, the subgroup $\mathbb{Z}_{k_o}$ acts trivially on V. Define the new action of $S^1 \simeq S^1/\mathbb{Z}_{k_o}$ on the space V, which is also an orthogonal S^1-representation, and denote this new representation by $\tilde{V}$. Moreover, the map f remains S^1-equivariant with respect to this new action. Denote by $\tilde{\Omega}$ the set Ω considered as an S^1-subspace of $\tilde{V}$. Consequently, (f, Ω) is the k_o-folded admissible pair associated with the admissible pair $(f, \tilde{\Omega})$. By the folding property (F), we have

$$S^1\text{-Deg}\,(f, \Omega) = \Theta_{k_o}\left[S^1\text{-Deg}\,(f, \tilde{\Omega})\right].$$

To conclude the argument, it is sufficient to show that

$$S^1\text{-Deg}\,(f, \tilde{\Omega}) = n_o\,(\mathbb{Z}_1).$$

In the remaining part of the proof, we will assume that $G_{x_o} = \mathbb{Z}_1$.

Step 2: Reduction to a tubular neighborhood.

Take a tubular neighborhood Ω' around the orbit $G(x_o)$, i.e.

$$\Omega' = G\big(x_o + B_\varepsilon(S_{x_o})\big), \quad 0 < \varepsilon < \|x_o\|,$$

where S_{x_o} is the positively oriented slice to the orbit $G(x_o)$ at x_o. Then every point $x \in \Omega'$ has a unique representation as $\gamma x_o + \gamma v$, for some $v \in B_\varepsilon(S_{x_o})$ and $\gamma \in S^1$.

Define the linear operator

$$A := Df(x_o)_{|S_{x_o}} : S_{x_o} \to V,$$

and the map $f_0 := \overline{\Omega'} \to V$ by

$$f_0\big(\gamma(x_o + v)\big) = \gamma(Av), \quad \gamma \in S^1, \quad v \in B_\varepsilon(S_{x_o}),$$

which is clearly S^1-equivariant. By the excision property (P7') and homotopy property (P3), we have that

$$S^1\text{-Deg}\,(f_0, \Omega') = S^1\text{-Deg}\,(f, \Omega).$$

Step 3: Reduction to One Isotypical Component.

We consider the path $x_\lambda = \lambda e + (1-\lambda)x_o$, $\lambda \in [0,1]$, where e is a unit vector belonging to the isotypical component V_1. Let S_{x_λ} be the slice to the orbit $G(x_\lambda)$ at the point x_λ, and $B_\lambda = \{v \in S_{x_\lambda} : \|v\| < \varepsilon\}$ for $\min\{\|x_o\|, 1\} > \varepsilon > 0$. We put $\Omega_\lambda := G(x_\lambda + B_\lambda)$, $A_\lambda := Df(x_\lambda)_{|S_{x_\lambda}}$ and define $f_\lambda : \overline{\Omega_\lambda} \to V$, $\lambda \in [0,1]$, by

$$f_\lambda(\gamma(x_\lambda + v)) = \gamma(A_\lambda v), \quad v \in S_{x_\lambda}, \quad \gamma \in S^1.$$

By the excision property (P7)' and the homotopy property (P3), we have

$$S^1\text{-Deg}\,(f_1, \Omega_1) = S^1\text{-Deg}\,(f_\lambda, \Omega_\lambda) = S^1\text{-Deg}\,(f_0, \Omega') = S^1\text{-Deg}\,(f, \Omega).$$

Notice that, using a path in the space of linear isomorphisms from S_e to V, the matrix A can be deformed to a block matrix $\tilde{A}$, which is Id on the isotypical components $V_{k_2}, \ldots, V_{k_r}$. By the suspension property (P4), we can assume that $V = V^G \oplus V_1$, $e \in V_1$.

Step 4: Reduction to Basic Maps.

Suppose that $V_1 = \mathbb{C}^k = \mathbb{R}^{2k}$ and $e = (0,0,\ldots,0,1,0)$. Since the orbit $G(e)$ consists of the points $(0,\, 0,\, \ldots,\, 0,\, \cos\tau,\, \sin\tau) \in \mathbb{R}^{2k}$, the tangent vector to $G(e)$ at e is the vector $v_{2k+1} = (0,0,\ldots,0,1)$, and consequently the slice S_e consists of all vectors of the form $(\alpha_1, \alpha_2, \ldots, \alpha_{2k-1}, 0)$, $\alpha_j \in \mathbb{R}$. By taking the standard basis in S_e, which in this case defines the positive orientation of

S_e, we can use the fact that there exists a path A_λ ($\lambda \in [0,1]$), in $GL(2k,\mathbb{R})$ connecting the matrix $\tilde{A}$ to the matrix:

$$A_1 := \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

if sign det $Df(x_o)|_{S_{x_o}} > 0$, and

$$A_1 := \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

if sign det $Df(x_o)|_{S_{x_o}} < 0$. The path A_λ defines an Ω_1-admissible homotopy

$$f_{1+\lambda}\big(\gamma(e+v)\big) = \gamma(A_\lambda v), \quad v \in S_e, \quad \gamma \in S^1.$$

Let us consider an element $(t,v) \in \mathbb{R} \oplus V$, which is represented as

$$(t,v) = v_0 + \tilde{v}_1 + \gamma s e, \quad v_0 \in V^G, \;\; \tilde{v}_1 \in \mathbb{C}^{k-1} \times \{0\} \subset \mathbb{C}^k = V_1, \; \gamma \in S^1, \; s \in \mathbb{R}_+.$$

Then we have

$$\begin{aligned} f_2(t,v) &= f_2(t, v_0 + \tilde{v}_1 + \gamma s e) = f_2(\gamma(t, v_0 + \gamma^{-1}\tilde{v}_1 + s e) \\ &= \gamma\big(A_1(t, v_0 + \gamma^{-1}\tilde{v}_1 + s e\big) = \gamma(v_0 + \gamma^{-1}\tilde{v}_1) + \gamma\tilde{A}_1(t,s) \\ &= v_0 + \tilde{v}_1 + \gamma\tilde{A}_1(t,s), \end{aligned}$$

where $\tilde{A}_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ if sign det $Df(x_o)|_{S_{x_o}} > 0$ and $\tilde{A}_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ if sign det $Df(x_o)|_{S_{x_o}} < 0$. The above identities show that the map f_2 is "normal" with respect to the vectors $v_0 + \tilde{v}_1$, i.e. $f_2 = \tilde{f}_2 \times \mathrm{Id}$, where $\tilde{f}_2 : \mathbb{R} \oplus \mathbb{C} \to \mathbb{C}$ is given by:

$$\tilde{f}_2(t, \gamma s e) = \gamma\big(\tilde{A}_1(t,s)\big), \quad \gamma \in S^1, \; s \in \mathbb{R}_+, \; t \in \mathbb{R}.$$

Therefore, by the suspension property (P4), we have

$$S^1\text{-Deg}\,(f_2, \Omega_1) = S^1\text{-Deg}\,(\tilde{f}_2, \tilde{\Omega}_1),$$

where $\tilde{\Omega}_1 := \left\{(t,z) \in \mathbb{R} \oplus \mathbb{C} : |t| < 1,\ \frac{1}{2} < |z| < 2\right\}$ is equivariantly homotopically equivalent to Ω_1, and the S^1-action on $\mathbb{C}$ is the standard complex multiplication.

Let us consider the maps $b(t,z) = \big(1-|z|+it\big)\cdot z$ and $b^-(t,z) = \big(1-|z|-it\big)\cdot z$, defined on $\tilde{\Omega}_1$, to which we can apply the linearization procedure along the orbit $G(z_o)$, $z_o = (0,1,0) \in \mathbb{R} \oplus \mathbb{C}$. More precisely, we consider the derivatives $Db(0,1,0)$ and $Db^-(0,1,0)$ restricted to S_e, which can be easily evaluated:

$$B_+ := Db(0,1,0)_{|S_e} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \quad B_- := Db^-(0,1,0)_{|S_e} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{3.21}$$

Then, by applying the formula $f_\pm(t,\gamma s) := \gamma(B_\pm(t,s))$, $\gamma \in S^1$, $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$, we observe that f_+ (resp. f_-) is equivariantly homotopic to the basic map b (resp. b^-). Therefore, if sign det $Df(x_o)|_{S_{x_o}} = 1$, then there exists an $\tilde{\Omega}_1$-admissible homotopy between b and $\tilde{f}_2$, and if sign det $D_{S_{x_o}} f(x_o) = -1$, then there exists an $\tilde{\Omega}_1$-admissible homotopy between b^- and $\tilde{f}_2$. Consequently, by the normalization property (P5) and Corollary 3.4.7, we obtain that

$$S^1\text{-Deg}\,(f,\Omega) = n_o\,(\mathbb{Z}_1),$$

which completes the proof. □

Proof of Theorem 3.4.4

Existence. We claim that the primary degree defined by the formulae (3.4)—(3.6) (with $n = 1$ and $G = S^1$) satisfies the properties listed in Theorem 3.4.4. Indeed, Properties (P1)—(P4), (P6)' are provided by Proposition 3.2.4. Property (P5)' follows from (3.21). To show (F), consider an admissible pair (f,Ω) and the associated m-folded pair $(f, {}^m(\Omega))$. By the homotopy and excision properties, we can assume that f is regular normal on Ω (and, consequently, on ${}^m(\Omega)$). Take some orbit type $(\mathbb{Z}_k)$ occuring in Ω and let D be a regular fundamental domain for $\Omega_{\mathbb{Z}_k}$. Then D is a regular fundamental domain for ${}^m(\Omega)_{\mathbb{Z}_{km}}$. Since f is the same for both cases, the result follows from (3.5).

Uniqueness. Let $\widetilde{S^1\text{-Deg}}$ be a function satisfying Properties (P1)—(P4), (P5)', (P6)' and (F). Let V be an orthogonal S^1-representation, $\Omega \subset \mathbb{R} \oplus V$ an S^1-invariant open bounded region, and $f : \mathbb{R} \oplus V \to V$ an equivariant Ω-admissible map. We will show that

$$\widetilde{S^1\text{-Deg}}\,(f,\Omega) = S^1\text{-Deg}\,(f,\Omega).$$

By Proposition 2.3.5 and homotopy property (P3), without lost of generality one can assume that f is regular normal. By the normality, there exists an open S^1-invariant subset $\Omega_o \subset \Omega$ such that $Z := f^{-1}(0) \cap \Omega^{S^1} = f^{-1}(0) \cap \Omega_o$, i.e. Ω_o is an *isolating invariant neighborhood* of Z. In addition, we can assume that $f_{|\Omega_o}$ (up to an Ω_o-admissible homotopy) is a product map $f^{S^1} \times \mathrm{Id}$, where $f^{S^1} := f_{|\mathbb{R}\oplus V^{S^1}}$, and Id is the identity operator on the space $(\mathbb{R} \oplus V^{S^1})^{\perp}$. Then, by the suspension property (P4) and the elimination property (P6)', we have

$$S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_o) = S^1\text{-}\widetilde{\mathrm{Deg}}\,(f^{S^1} \times \mathrm{Id}, \Omega_o^{S^1} \times B) = S^1\text{-}\widetilde{\mathrm{Deg}}\,(f^{S^1}, \Omega_o^{S^1}) = 0,$$

where B denotes the unit ball in $(\mathbb{R} \oplus V^{S^1})^{\perp}$.

Since f is assumed to be regular, we have that

$$f^{-1}(0) \cap \Omega = Z \cup S^1(x_1) \cup \cdots \cup S^1(x_m),$$

where $S^1(x_j)$, $j = 1, 2, \ldots, m$, are isolated orbits. We can choose open invariant sets $\Omega_j \subset \Omega$ such that $\Omega_j \supset S^1(x_j)$, $\Omega_j \cap \Omega_i = \emptyset$, $i \neq j$, $i, j = 0, 1, 2, \ldots, m$. Then, by applying the additivity property (P2), we obtain that

$$\begin{aligned} S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega) &= S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_0) + S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_1) + \cdots + S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_m) \\ &= S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_1) + \cdots + S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_m). \end{aligned}$$

For each of the orbits $S^1(x_j)$, $j = 1, \ldots, m$, we consider the positively oriented slice S_j at the point x_j, and we denote by $D_j f(x_j)$ the matrix of the derivative $Df(x_j)_{|S_j}$, with respect to a basis in S_j defining the positive orientation on it.

Applying Lemma 3.4.5 and Properties (P2), (P7)', one obtains

$$\begin{aligned} S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega) &= \sum_{j=1}^{m} S^1\text{-}\widetilde{\mathrm{Deg}}\,(f, \Omega_j) = \sum_{j=1}^{m} \operatorname{sign} \det Df(x_j)|_{S_j} \cdot (\mathbb{Z}_{k_j}) \\ &= \sum_{j=1}^{m} S^1\text{-}\mathrm{Deg}\,(f, \Omega_j) = S^1\text{-}\mathrm{Deg}\,(f, \Omega). \end{aligned}$$

□

We present some immediate consequences of Theorem 3.4.4.

Corollary 3.4.6. *The S^1-degree provided by Theorem 3.4.4 also satisfies*

(P7)' (EXCISION) *Assume Ω_o is an S^1-invariant open subset of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_o$. Then,*

$$S^1\text{-Deg}\,(f, \Omega) = S^1\text{-Deg}\,(f, \Omega_o).$$

(P9) (l-TH BASIC MAP) *For every $l = 1, 2, 3, \ldots$ and the basic map b associated with l-th irreducible S^1-representation, we have (cf. Notation 3.4.2)*

$$S^1\text{-Deg}\,(b, {}^l\Omega) = (\mathbb{Z}_l).$$

The proof of Corollary 3.4.6 is straightforward and we omit it.

Corollary 3.4.7. *Let $b^- : \mathbb{R} \oplus {}^l\mathcal{V} \to {}^l\mathcal{V}$, $l = 1, 2, 3, \ldots$, be defined by*

$$b^-(t, z) = \big(1 - |z| - it\big) \cdot z, \quad t \in \mathbb{R}, \ z \in {}^l\mathcal{V}. \tag{3.22}$$

Then,

$$S^1\text{-Deg}\,(b^-, {}^l\Omega) = -(\mathbb{Z}_l). \tag{3.23}$$

Proof: We consider the set

$$\Omega := \left\{(t, z) \in \mathbb{R} \oplus {}^l\mathcal{V} : |t| < 2, \ \frac{1}{2} < |z| < 2\right\}$$

and the function $\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$\alpha(t) = \begin{cases} 1 & \text{if } \ t < -1 \ \text{ or } \ t > \frac{3}{2}, \\ -t & \text{if } \ -1 \le t < \frac{1}{4}, \\ t - \frac{1}{2} & \text{if } \ \frac{1}{4} \le t \le \frac{3}{2}. \end{cases}$$

Define the homotopy $h : [0, 1] \times \mathbb{R} \oplus {}^l\mathcal{V} \to {}^l\mathcal{V}$ by

$$h_\lambda(t, z) = \Big(\lambda(1 - |z|) + i\big((1 - \lambda) + \lambda\alpha(t)\big)\Big) \cdot z, \quad z \in {}^l\mathcal{V}, \ t \in \mathbb{R}, \ \lambda \in [0, 1].$$

It is clear that h_λ is an Ω-admissible homotopy such that $h_0(t, z) = i \cdot z$, which implies (by (P1)) that $S^1\text{-Deg}\,(h_0, \Omega) = 0$ and, therefore (by (P3)),

$$S^1\text{-Deg}\,(h_1, \Omega) = 0. \tag{3.24}$$

Obviously, $h_1^{-1}(0) \cap \Omega = \left\{(t, z) \in \mathbb{R} \oplus {}^l\mathcal{V} : |z| = 1, \ t = 0, \ \frac{1}{2}\right\}$. Put

$$\Omega_1 := \left\{(t,z) \in \mathbb{R} \oplus {}^l\mathcal{V} : |t| < \frac{1}{4},\ \frac{1}{2} < |z| < 2\right\},$$

$$\Omega_2 := \left\{(t,z) : \left|t - \frac{1}{2}\right| < \frac{1}{4},\ \frac{1}{2} < |z| < 2\right\}.$$

Then (by (P2) and (3.24))

$$S^1\text{-Deg}\,(h_1, \Omega_1) + S^1\text{-Deg}\,(h_1, \Omega_2) = 0. \tag{3.25}$$

By (P7)' (resp. (P3)), we have

$$S^1\text{-Deg}\,(h_1, \Omega_1) = S^1\text{-Deg}\,(b^-, {}^l\Omega) \quad \left(\text{resp.} \quad S^1\text{-Deg}\,(h_1, \Omega_2) = S^1\text{-Deg}\,(b, {}^l\Omega)\right).$$

Therefore, by (P9) and (3.25), $S^1\text{-Deg}\,(b^-, {}^l\Omega) = -(\mathbb{Z}_l)$. □

3.4.2 Computational Formulae for S^1-Degree

Based on the S^1-degrees of basic maps, by Proposition 3.3.1, we obtain similar result for $\mathbb{C}$-complementing maps (cf. Definition 3.3.2).

Corollary 3.4.8. *(i) Let $(f, {}^l\mathcal{O})$ be a $\mathbb{C}$-complementing pair to $(b, {}^l\Omega)$. Then, f is S^1-homotopic to a map $\overline{f_1}$, which is a suspension of b on an open subset containing zeros of $\overline{f_1}$. In particular,*

$$S^1\text{-Deg}\,(f, {}^l\mathcal{O}) = S^1\text{-Deg}\,(b, {}^l\Omega) = (\mathbb{Z}_l). \tag{3.26}$$

(ii) Similarly, let $(f^-, {}^l\mathcal{O})$ be a $\mathbb{C}$-complementing pair to $(b^-, {}^l\Omega)$. Then, f^- is S^1-homotopic to a map $\overline{f_1^-}$, which is a suspension of b^- on an open subset containing zeros of $\overline{f_1^-}$. Moreover,

$$S^1\text{-Deg}\,(f^-, {}^l\mathcal{O}) = S^1\text{-Deg}\,(b^-, {}^l\Omega) = -(\mathbb{Z}_l). \tag{3.27}$$

As consequence of Splitting Lemma (cf. Lemma 3.3.4), we have the following computational formulae of the S^1-degree.

Corollary 3.4.9. *Let ${}^l\mathcal{V}$ be the l-th irreducible S^1-representation. Define*

$$\tilde{f}(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, \left(\frac{\lambda}{|\lambda|}\right)^k v\right), \quad (\lambda, v) \in \overline{\mathcal{O}},$$

where ${}^l\mathcal{O}$ is given by (3.9) (cf. Notation 3.4.2). Then, $S^1\text{-Deg}\,(\tilde{f}, {}^l\mathcal{O}) = k(\mathbb{Z}_l)$.

Proof: For the sake of definiteness, assume that $k > 0$ (the case $k \leq 0$ can be treated using a similar argument), and consider the map

$$\tilde{f} \times \mathrm{Id} : \overline{{}^l\mathcal{O}} \times \overline{B_{k-1}} \to \mathbb{R} \oplus {}^l\mathcal{V}_l \oplus \Big[\underbrace{{}^l\mathcal{V} \oplus \cdots \oplus {}^l\mathcal{V}}_{k-1}\Big],$$

where $B_{k-1} = \underbrace{B({}^l\mathcal{V}) \times \cdots \times B({}^l\mathcal{V})}_{k-1}$ and $B({}^l\mathcal{V})$ denotes the unit ball in ${}^l\mathcal{V}$. Then, by suspension property,

$$S^1\text{-Deg}\,(\tilde{f},\, {}^l\mathcal{O}) = S^1\text{-Deg}\left(\tilde{f} \times \mathrm{Id},\, {}^l\mathcal{O} \times B_{k-1}\right).$$

Obviously, $\tilde{f} \times \mathrm{Id}$ is equivariantly homotopic, by an ${}^l\mathcal{O} \times B_{k-1}$-admissible homotopy, to f_a given by (3.11), where $v \in V = \underbrace{{}^l\mathcal{V} \oplus \cdots \oplus {}^l\mathcal{V}}_{k}$ and $a : S^1 \to GL^{S^1}(V)$ is defined by

$$a(\gamma) = \begin{bmatrix} \gamma^k & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \gamma \in S^1.$$

By an ${}^l\mathcal{O} \times B_{k-1}$-admissible homotopy, f_a is equivariantly homotopic to f_b given by

$$f_b(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, b\left(\frac{\lambda}{|\lambda|}\right) v\right),$$

with $b : S^1 \to GL^{S^1}(V)$ defined by

$$b(\gamma) = \begin{bmatrix} \gamma & 0 & \dots & 0 \\ 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma \end{bmatrix}, \quad \gamma \in S^1.$$

Since $S^1\text{-Deg}\,(\tilde{f},\, {}^l\mathcal{O}) = S^1\text{-Deg}\,(f_b,\, {}^l\mathcal{O} \times B_{k-1})$, by the Splitting Lemma and Proposition 3.3.1, we have

$$S^1\text{-Deg}\,(\tilde{f},\, {}^l\mathcal{O}) = \underbrace{(\mathbb{Z}_l) + \cdots + (\mathbb{Z}_l)}_{k} = k(\mathbb{Z}_l).$$

The proof of Corollary 3.4.9 is complete. □

By combining the Splitting Lemma and Corollary 3.4.9, we obtain

Theorem 3.4.10. *Let V be an orthogonal S^1-representation with $V^{S^1} = \{0\}$, admitting the isotypical decomposition (2.5). Let $\mathcal{O}$ (resp. f_a) be defined by (3.9) (resp. (3.11)). Then*

$$S^1\text{-Deg}\,(f_a, \mathcal{O}) = \sum_{i=1}^{s} k_i(\mathbb{Z}_{l_i}),$$

where $k_i := \deg\left(\det_{\mathbb{C}} \circ a_i, S^1\right), \quad a_i(\lambda) := a(\lambda)|_{V_{l_i}} : V_{l_i} \to V_{l_i}$*, for* $i = 1, \dots, r$.

As an immediate consequence of Theorem 3.4.10, we obtain

Corollary 3.4.11. *Let V and $\mathcal{O}$ be as in Theorem 3.4.10. Let $k_i \in \mathbb{Z}$, $i = 1, \dots, r$, be given integers and assume that* $\dim_{\mathbb{C}} V_{l_i} = m_i$. *Define $f : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by* $f(\lambda, v_1, \dots, v_s) = \Big(|\lambda|\big(\|v\| - 1\big) + \|v\| + 1, \lambda^{k_1} v_1, \dots, \lambda^{k_s} v_s\Big)$, *where* $\lambda \in \mathbb{C} \setminus \{0\}$, $v_i \in V_{l_i}$. *Then*

$$S^1\text{-Deg}\,(f, \mathcal{O}) = \sum_{i=1}^{s} m_i k_i(\mathbb{Z}_{l_i}).$$

3.5 Recurrence Formulae

In this section, we present two recurrence formulae. The first one allows us to reduce the computation of the primary G-degree of one parameter to the computations of the related S^1-degrees, while the second one facilitates the computations of primary G-degree without free parameters.

3.5.1 One Parameter Case

To formulate this formula, we need the following notations.

Notation 3.5.1 Let V be an orthogonal S^1-representation, $\Omega \subset \mathbb{R} \oplus V$ an open bounded S^1-invariant set, and $f : \mathbb{R} \oplus V \to V$ an Ω-admissible S^1-equivariant map. Consider the S^1-degree defined by (3.18) and put

$$\deg_{k_i}(f, \Omega) := n_{k_i}, \quad i = 1, 2, \dots, r.$$

Observe that each of the integer coefficients n_{k_i} satisfies the usual additivity, homotopy, excision, and suspension properties.

For simplicity, we assume that

(*) for all $(H), (K), (L) \in \Phi_1(G)$ with $(H) < (K) < (L)$ and $(H), (L) \in \Phi_1^+(G)$, we have that $(K) \in \Phi_1^+(G)$.

Remark 3.5.2. In all computational examples considered in this thesis, the assumption (*) automatically verifies. In the general case, one needs to extend the notion of the primary degree to include the relatively bi-orientable orbit types and similar statement holds [12, 15].

Proposition 3.5.3. (Recurrence Formula) *Let V be an orthogonal G-representation, $\Omega \subset \mathbb{R} \oplus V$ an open bounded invariant subset and $f : \mathbb{R} \oplus V \to V$ a G-equivariant Ω-admissible map. Under the assumption (*), we have that*

$$G\text{-Deg}\,(f, \Omega) = \sum_{(H) \in \Phi_1^+(G)} n_H \cdot (H),$$

where

$$n_H = \left[\sum_k \deg_k(f^H, \Omega^H) - \sum_{(H_o) > (H)} n_{H_o} n(H, H_o)\, |W(H_o)/S^1| \right] \Big/ |W(H)/S^1|.$$

Notice that a particular case of Proposition 3.5.3 was established in [114], where the argument is based on the using the S^1-fixed point index.

Proof: By the definition of n_H, it is an algebraic count of $W(H)$-orbits in Ω_H. Since $\dim W(H) = 1$, it is diffeomorphic to a disjoint union of m copies of S^1, where $m = |W(H)/S^1|$. In another word, $n_H \cdot |W(H)/S^1|$ gives an algebraic count of S^1-orbits in Ω_H.

Observe that $\Omega^H = \bigcup_{H_o \supseteq H} \Omega_{H_o}$. To count the S^1-orbits in Ω_H, it is sufficient to do the counting first in Ω^H, then subtract off those S^1-orbits belonging to Ω_{H_o} for $H_o \supset H$. In order to count the S^1-orbits in Ω^H, it is sufficient to compute the value of S^1-Deg$\,(f^H, \Omega^H)$, then sum up the coefficients related to $(\mathbb{Z}_k)$ for all $k \in \mathbb{N}$, i.e. it equals to $\sum_k \deg_k(f^H, \Omega^H)$. On the other hand, $n_{H_o} \cdot |W(H_o)/S^1|$ represents the count of S^1-orbits in Ω_{H_o}.

Therefore, we have

$$\begin{aligned} n_H \cdot |W(H)/S^1| &= \sum_k \deg_k(f^H, \Omega^H) - \sum_{H_o \supset H} n_{H_o} \cdot |W(H_o)/S^1| \\ &= \sum_k \deg_k(f^H, \Omega^H) - \sum_{(H_o)>(H)} n(H, H_o) n_{H_o} \cdot |W(H_o)/S^1|, \end{aligned}$$

which completes the proof.

□

We provide an example of computation for a primary $D_6 \times S^1$-degree using the recurrence formula. The conventions of notations follow Appendix A2.

Example 3.5.4. Let $G = D_6 \times S^1$ and take an irreducible G-representation $\mathcal{V}_{2,1} \simeq \mathbb{C} \oplus \mathbb{C}$ with the G-action given by

$$\begin{aligned} (\gamma, e^{i\theta})(z, w) &:= e^{i\theta} \cdot (\gamma^2 \cdot z, \gamma^{-2} \cdot w), \\ (\kappa, e^{i\theta})(z, w) &:= e^{i\theta} \cdot (w, z), \end{aligned}$$

where $\gamma^6 = 1$ and $e^{i\theta} \in S^1$. Let $b : \mathbb{R} \oplus \mathcal{V}_{2,1} \to \mathcal{V}_{2,1}$ be the basic map associated to $\mathcal{V}_{2,1}$ given by

$$b(t, v) := \big(1 - \|v\| + it\big) \cdot v, \ t \in \mathbb{R}, \ v \in \mathcal{V}_{2,1},$$

and $\Omega := \{(t, v) \in \mathbb{R} \oplus \mathcal{V}_{2,1} \ : \ |t| < 1, \ \frac{1}{2} < \|v\| < 2\}$ (cf. Definition 3.3.2).

To evaluate $G\text{-Deg}\,(b, \Omega)$, we use an induction over the lattice of the orbit types according to the recurrence formula. Since the orbit types occuring in $\mathcal{V}_{2,1}$ are $(\mathbb{Z}_6^{t_2})$, (D_2^z), (D_2) and $(\mathbb{Z}_2)$, we suppose

$$G\text{-Deg}\,(b, \Omega) = n_1(\mathbb{Z}_6^{t_2}) + n_2(D_2^z) + n_3(D_2) + n_4(\mathbb{Z}_2),$$

for integers $n_i \in \mathbb{Z}$, $i = 1, 2, 3, 4$. Following the lattice of these four orbit types (cf. Figure A2.12 with $N = 6$, $j = 2$ and $h = 2$), we first compute the coefficients n_1, n_2, n_3 for the maximal orbit types $(\mathbb{Z}_6^{t_2})$, (D_2^z), (D_2) respectively. Using the maximality of $(\mathbb{Z}_6^{t_2})$ and the fact that $\dim_{\mathbb{C}}(\Omega^{\mathbb{Z}_6^{t_2}}) = 1$, we have that $S^1\text{-Deg}\,(b^{\mathbb{Z}_6^{t_2}}, \Omega^{\mathbb{Z}_6^{t_2}}) = 1 \cdot (\mathbb{Z}_1)$. Taking into account $W(\mathbb{Z}_6^{t_2}) = \mathbb{Z}_1 \times S^1$, we then have

$$\begin{aligned} n_1 &= \deg_1(b^{\mathbb{Z}_6^{t_2}}, \Omega^{\mathbb{Z}_6^{t_2}})/|W(\mathbb{Z}_6^{t_2})/S^1| \\ &= 1/|\mathbb{Z}_1 \times S^1/S^1| \\ &= 1. \end{aligned}$$

Similarly, we obtain that $n_2 = n_3 = 1$. To compute the coefficient n_4 for the orbit type $(\mathbb{Z}_2)$, observe that $\Omega^{\mathbb{Z}_2} = \Omega$. Thus, $S^1\text{-Deg}\,(b^{\mathbb{Z}_2}, \Omega^{\mathbb{Z}_2}) = S^1\text{-Deg}\,(b, \Omega)$. By the splitting lemma, $S^1\text{-Deg}\,(b, \Omega) = m \cdot (\mathbb{Z}_1)$, where $m = \dim_{\mathbb{C}} \Omega = 2$. Therefore,

$$\begin{aligned} n_4 &= (\deg_1(b^{\mathbb{Z}_2}, \Omega^{\mathbb{Z}_2}) - n_1 \cdot N(\mathbb{Z}_2, \mathbb{Z}_6^{t_2})|W(\mathbb{Z}_6^{t_2})/S^1| - n_2 \cdot N(\mathbb{Z}_2, D_2^z)|W(D_2^z)/S^1| \\ &\quad - n_3 \cdot N(\mathbb{Z}_2, D_2)|W(D_2)/S^1|)/|W(\mathbb{Z}_2)/S^1| \\ &= (2 - 1 \cdot 2 \cdot 1 - 1 \cdot 3 \cdot 1 - 1 \cdot 3 \cdot 1)/6 \\ &= -1, \end{aligned}$$

where we use the facts $N(\mathbb{Z}_2, \mathbb{Z}_6^{t_2}) = 2$, $N(\mathbb{Z}_2, D_2^z) = N(\mathbb{Z}_2, D_2) = 3$, $W(\mathbb{Z}_6^{t_2}) = W(D_2^z) = W(D_2) = \mathbb{Z}_1 \times S^1$ and $W(\mathbb{Z}_2) = D_3 \times S^1$.

Consequently, $G\text{-Deg}\,(b, \Omega) = (\mathbb{Z}_6^{t_2}) + (D_2^z) + (D_2) - (\mathbb{Z}_2)$. In fact, we just computed the so-called twisted basic degree of $\mathcal{V}_{2,1}$ (cf. Definition 4.2.8).

3.5.2 No Parameters Case

Following the same idea as the proof of Proposition 3.5.3, we obtain (cf. [116, 5, 114, 47])

Proposition 3.5.5. (Recurrence Formula) *Let V be an orthogonal G-representation, $\Omega \subset V$ an open bounded invariant subset and $f : V \to V$ a G-equivariant Ω-admissible map. Then, we have that*

$$G\text{-Deg}\,(f, \Omega) = \sum_{(H) \in \Phi_0(G)} n_H \cdot (H),$$

where

$$n_H = \left[\deg(f^H, \Omega^H) - \sum_{(H_o) > (H)} n_{H_o} n(H, H_o)\, |W(H_o)| \right] \Big/ |W(H)|.$$

As an illustration of the usage of the above recurrence formula, we compute a primary D_6-degree without parameters. For the conventions of notations, we refer to Appendix A2.

Example 3.5.6. Let $G = D_6$ and take an irreducible G-representation $\mathcal{V}_4 \simeq \mathbb{R}$, which is induced by the homomorphism $\varphi : D_6 \to \mathbb{Z}_2$ with $\ker \varphi = D_3$. Consider the basic map $b := -\text{Id} : \mathcal{V}_4 \to \mathcal{V}_4$ on the unit ball $B_1 := B_1(\mathcal{V}_4)$

and compute $G\text{-Deg}(-\mathrm{Id}, B_1)$. Observe that the only orbit types occuring in $\mathcal{V}_4$ are (D_6) and (D_3). Suppose that

$$G\text{-Deg}(b, B_1) = n_1(D_6) + n_2(D_3).$$

Since $B_1^{D_6} = \{0\}$, we have that $\deg(b^{D_6}, B_1^{D_6}) = 1$. Thus

$$n_1 = \deg(b^{D_6}, \Omega^{D_6})/|W(D_6)| = 1.$$

Similarly, from $B_1^{D_3} \simeq \mathbb{R}$, it follows that $\deg(b^{D_3}, B_1^{D_3}) = -1$. Therefore,

$$\begin{aligned} n_2 &= \big(\deg(b^{D_3}, B_1^{D_3}) - n_1 \cdot N(D_3, D_6)|W(D_6)|\big)/|W(D_3)| \\ &= (-1 - 1 \cdot 1 \cdot 1)/2 \\ &= -1. \end{aligned}$$

Consequently, $G\text{-Deg}(b, B_1) = (D_6) - (D_3)$, which is indeed the so-called basic degree without parameters associated to $\mathcal{V}_4$ (cf. Definition 4.1.5).

4

Twisted Primary Degree

In this chapter, we assume that $G = \Gamma \times S^1$ for Γ being a compact Lie group. We introduce the so-called *twisted equivariant degree*, which is defined as a truncated part of the primary G-equivariant degree with one free parameter. The twisted equivariant degree turns out to be the most "computable" part of the primary equivariant degree, and thus serves as an effective topological tool in the study of various applied problems, including the Γ-symmetric Hopf bifurcation problems and the existence of periodic solutions in autonomous systems (cf. Part II).

Among the important "predecessors" of the (twisted) S^1-equivariant degree, one should mention the rational-valued homotopy invariants introduced and studied in [67, 40, 42, 44, 43].

The effective usage of the twisted equivaraint degree method highly depends on an important property called the *multiplicativity* property, which is analogous to the multiplicativity property of the classical Brouwer degree taken in the integer ring $\mathbb{Z}$. In the case of the primary degree without free parameters, this property is related to a natural ring structure of its range $A_0(\Gamma)$, called the *Burnside ring*. In the case of the twisted equivariant degree, it takes a form of a $A_0(\Gamma)$-module multiplication in the range $A_1^t(\Gamma \times S^1)$. The multiplication in both cases expresses the orbit structure in a Cartesian product of two orbits. In general, explicit multiplication tables for an arbitrary compact Lie group Γ are difficult to establish. Nevertheless, based on certain recurrence formulae, a series of examples of the multiplication tables are obtained and listed in the Appendix A3 for Γ equal to the quaternionic group Q_8, dihedral group D_N, symmetry groups A_4, S_4, A_5 and orthogonal group $O(2)$.

By the multiplicity property, the computations of the twisted equivariant degree can be significantly reduced to the evaluations of the twisted degrees of the basic maps. Since the twisted degrees of basic maps (called *basic degrees*), stand out of context of any specific applied scheme and depend only on the group Γ and its irreducible representations, the values of the basic degrees can be computed systematically in advance, using certain recurrence formulae, and simply included as a part of the database for the equivariant degree methods. See Appendix A3 for examples of basic degrees, where $\Gamma = Q_8, D_N, A_4, S_4, A_5, O(2)$.

This chapter is organized as follows. In Section 4.1, we recall the Burnside ring structure on $A_0(\Gamma)$ and provide the recurrence formula for the multiplication operation. Also, we define the *basic degrees* in the setting of the primary degree without parameters and present the corresponding recurrence formula. In Section 4.2, we introduce the *twisted subgroups* of $G = \Gamma \times S^1$ and define $A_1^t(G) \subset A_1^+(G)$ as a $\mathbb{Z}$-submodule generated by the conjugacy classes of the twisted subgroups in G. This submodule $A_1^t(G)$ has an $A_0(\Gamma)$-module structure, which can be determined by a recurrence formula. We also define the *twisted primary degree* for $G = \Gamma \times S^1$, as a truncated primary degree, taking values in $A_1^t(G)$. Finally, the *twisted basic degree* will be introduced, which plays an important role in obtaining all the computational results presented in this thesis.

4.1 Burnside Ring and Basic Degrees without Free Parameter

4.1.1 Burnside Ring

Recall that $\Phi_0(\Gamma) = \{(H) \ : \ \dim W(H) = 0$. Denote by $A_0(\Gamma)$ the free abelian group generated by $\Phi_0(\Gamma)$. In order to define the multiplication operation on $A_0(\Gamma)$, observe that

$$\begin{aligned}(\Gamma/H \times \Gamma/K)_{(L)}/\Gamma &\cong (\Gamma/H \times \Gamma/K)_L/N(L)\\ &\subset (\Gamma/H \times \Gamma/K)^L/N(L)\\ &= (\Gamma/H^L \times \Gamma/K^L)/(N(L)/L)\\ &= (\Gamma/H^L \times \Gamma/K^L)/W(L).\end{aligned}$$

Since the spaces Γ/H^L and Γ/K^L consist of finitely many $N(L)/L$-orbits and by assumption, $N(L)/L$ is finite, Γ/H^L and Γ/K^L are also finite (cf. [116]). Consequently, the set $(\Gamma/H \times \Gamma/K)_{(L)}/\Gamma$ is finite.

Definition 4.1.1. Let Γ be a compact Lie group and $A_0(\Gamma)$ be the free abelian group generated by $\Phi_0(\Gamma)$. Define the *multiplication* on $A_0(\Gamma)$ by

$$(H)\cdot(K) = \sum_{(L)\in\Phi_0(\Gamma)} n_L(H,K)(L) \tag{4.1}$$

where $(H), (K), (L) \in \Phi_0(\Gamma)$ and $n_L(H,K)$ denotes the number of elements in the set $(\Gamma/H \times \Gamma/K)_{(L)}/\Gamma$, i.e.

$$n_L(H,K) := \left|(\Gamma/H \times \Gamma/K)_{(L)}/\Gamma\right|,$$

where $|X|$ denotes the number of elements in the set X. In other words, the number $n_L(H,K)$ represents the number of orbits of type (L) contained in the space $\Gamma/H \times \Gamma/K$. Equipped with the multiplication given by (4.1), $A_0(\Gamma)$ is called the *Burnside Ring* of Γ.

Notation 4.1.2 In the case $G = \Gamma \times S^1$, the Burnside ring $A_0(\Gamma)$ can be naturally identified with $A_0(G)$ by $(H) \mapsto (H \times S^1)$. Throughout the rest of this thesis, we will use this identification freely and possibly without further notice.

We refer to [116] for more details and proofs related to the above definition of the Burnside Ring.

Remark 4.1.3. (i) The computations of the multiplication table for $A_0(\Gamma)$ can be effectively conducted using a simple recurrence formula (cf. Proposition 3.5.5)

$$n_L(H,K) = \frac{n(L,H)|W(H)|n(L,K)|W(K)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L})n_{\widetilde{L}}|W(\widetilde{L})|}{|W(L)|}. \tag{4.2}$$

(ii) Examples of Burnside ring multiplication tables are provided in Appendix A3, for $\Gamma = Q_8, D_3, D_4, D_5, D_6, A_4, S_4, A_5, O(2)$.

4.1.2 Primary Degrees without Free Parameters

The Burnside ring structure naturally endows the primary equivariant degree without parameters, a multiplicativity property.

Proposition 4.1.4. *Let Γ be a compact Lie group and V_i be a Γ-orthogonal representation, for $i = 1, 2$. Assume that (f_i, Ω_i) is an admissible pair in V_i, for $i = 1, 2$. Then, we have*

(P7′) (Multiplicativity) *The product map* $f_1 \times f_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$ *is* $\Omega_1 \times \Omega_2$*-admissible, and*

$$\Gamma\text{-Deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = \Gamma\text{-Deg}(f_1, \Omega_1) \cdot \Gamma\text{-Deg}(f_2, \Omega_2),$$

where Γ-Deg *is the primary equivariant degree without free parameters and* '$\cdot$' *stands for the multiplication in the Burnside ring* $A_0(\Gamma)$.

4.1.3 Basic Degrees and Computational Formulae for Linear Isomorphisms

In the case of no-parameter equivariant maps, the concept of being the simplest possible equivariant maps having nontrivial degrees reduces to the $-\text{Id}$ map defined on a Γ-irreducible representation.

Definition 4.1.5. Let $\mathcal{V}$ be a real irreducible representation of Γ. Consider $-\text{Id} : \mathcal{V} \to \mathcal{V}$ and its primary equivariant degree (without free parameters) (cf. Proposition 3.2.4–3.2.5). We call $\deg_{\mathcal{V}} := G\text{-Deg}(-\text{Id}, B_1(\mathcal{V})) \in A_0(\Gamma \times S^1) \simeq A_0(\Gamma)$ the *basic degree* (without free parameters) of Γ associated to the irreducible representation $\mathcal{V}$.

Remark 4.1.6. (i) The computations of the basic degrees without free parameters can be achieved using the following recurrence formula (cf. [116]). Suppose that $\deg_{\mathcal{V}} = \sum_{(L)\in\Phi_0(G)} n_L(L)$. Then,

$$n_L = \frac{(-1)^{n_L} - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L}) \cdot n_{\widetilde{L}} \cdot |W(\widetilde{L})|}{|W(L)|}, \tag{4.3}$$

where $n_L = \dim \mathcal{V}^L$.

(ii) As examples, the basic degrees of $\Gamma = Q_8, D_N, A_4, S_4, A_5, O(2)$ are provided in Appendix A2.

It turns out that the computations of the primary degree without free parameters for general, usually nonlinear, Γ-maps, can often be reduced to the computations for symmetric linear isomorphisms $A : V \to V$, where V is a Γ-orthogonal representation. Based on the usage of the basic degrees and the multiplicativity property of the primary degree without free parameters (cf. Proposition 4.1.4), we derive a computational formula for $\Gamma\text{-Deg}(A, B_1(V)) \in A_0(G)$.

Since $A : V \to V$ is assumed to be a symmetric linear Γ-isomorphism, V allows a Γ-isotypical decomposition provided by the eigenspaces $E(\mu)$ of A, for $\mu \in \sigma(A)$, namely

$$V = \bigoplus_{\mu \in \sigma(A)} E(\mu).$$

By applying suspension property, $\Gamma\text{-Deg}(A, B_1(V))$ can be evaluated by $\Gamma\text{-Deg}(A, B_1(V^-))$, where $V^- \subset V$ is the maximal subspace on which A is negative definite. More precisely, let $\sigma_-(A)$ denote the negative spectrum of A. Then,

$$V^- = \bigoplus_{\mu \in \sigma_-(A)} E(\mu).$$

Moreover on V^-, A is homotopic to $-\text{Id}$, which can be viewed as a product map with respect to the above isotypical decomposition of V^-, by homotopy and multiplicativity properties, we have

$$\Gamma\text{-Deg}(A, B_1(V)) = \prod_{\mu \in \sigma_-(A)} \Gamma\text{-Deg}(-\text{Id}, B_1(E(\mu))).$$

A further reduction is possible, by viewing $E(\mu)$ as a Γ-invariant subspace in V and taking an isotypical decomposition

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where $E_i(\mu)$ is modeled on $\mathcal{V}_i$ for $i = 0, 1, \ldots, r$. Put

$$m_i(\mu) = \dim E_i(\mu)/\dim \mathcal{V}_i, \quad i = 0, 1, 2, \ldots, r. \tag{4.4}$$

and call it the $\mathcal{V}_i$-*multiplicity* of μ.

By applying the multiplicativity property, we obtain

$$\begin{aligned} \Gamma\text{-Deg}(A, B_1(V)) &= \prod_{\mu \in \sigma_-(A)} \prod_{i=0}^{r} (\Gamma\text{-Deg}(-\text{Id}, B_1(\mathcal{V}_i)))^{m_i(\mu)}, \\ &= \prod_{\mu \in \sigma_-(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}, \end{aligned} \tag{4.5}$$

where $m_i(\mu)$ is the $\mathcal{V}_i$-multiplicity of μ (cf. (4.4)) and we used the identification $A_0(\Gamma) \simeq A_0(G)$ (cf. Notation 4.1.2).

4.2 Twisted Primary Degree

4.2.1 Twisted Subgroups of $\Gamma \times S^1$

Let $G := \Gamma \times S^1$ for Γ being a compact Lie group.

Definition 4.2.1. For a subgroup $K \subset \Gamma$, a group homomorphism $\varphi : K \to S^1$ and an integer $l \in \mathbb{Z}$, define a *φ-twisted l-folded* subgroup by

$$K^{\varphi,l} := \{(\gamma, z) \in K \times S^1 \ : \ \varphi(\gamma) = z^l\}.$$

In the case $l = 1$, we use the notation K^{φ} and simply call it a *twisted* subgroup.

Remark 4.2.2. Notice that if $H = K^{\varphi,l}$ is a twisted subgroup and $(\widetilde{H}) \leq (H)$, then $\widetilde{H}$ is also a twisted subgroup. In particular, every subgroup $H_1 \in (H)$ is twisted. Consequently, it makes sense to talk about the lattice of the conjugacy classes of twisted subgroups in $\Gamma \times S^1$. Moreover, if $\dim W(K) = 0$ and $L^{\psi,m}$ is a twisted subgroup such that $(L^{\psi,m}) \geq (K^{\varphi,l})$, then by Lemma 2.4.5(i), we have $\dim W(L) = 0$ (where $W(K)$ and $W(L)$ are taken in Γ).

Denote by $\Phi_1^l(G)$ the set of all conjugacy classes of the twisted m-folded subgroups $H = K^{\varphi,l}$, $l = 1, 2, \dots$, such that $\dim W(H) = 1$. Let $A_1^l(G)$ be the free $\mathbb{Z}$-module generated by $\Phi_1^l(G)$.

We have the following

Proposition 4.2.3. *Let $H = K^{\varphi,l}$ be a twisted subgroup such that $(H) \in \Phi_1^l(G)$. Then, the Weyl group $W(H)$ of H in G is bi-orientable and can be equipped with the natural orientation induced from S^1.*

Proof: The twisted subgroup $H = K^{\varphi,l}$ is given by

$$K^{\varphi,l} := \Big\{(\gamma, z) \in K \times S^1 : \varphi(\gamma) = z^l\Big\},$$

and we have

$$W(H) = \frac{N_o \times S^1}{K^{\varphi,l}},$$

where $N_o = \{\gamma \in N(K) : \varphi(\gamma k \gamma^{-1}) = \varphi(k) \ \ \forall \ k \in K\}$. In order to prove that $W(H)$ is bi-orientable, it is sufficient to show that there exists a non-vanishing vector field $\widetilde{X} : W(H) \to \tau(W(H))$ which is invariant with respect to both left and right translations on $W(H)$. For this purpose, consider the vector field

$$X : N_o \times S^1 \to \tau(N_o \times S^1) = \tau(N_o) \times \tau(S^1),$$

defined by

$$X(\gamma, z) = \big((\gamma, z), v(z)\big),$$

where $v(z)$ is a unit tangent vector at z on S^1. More precisely, by using the identification

$$\tau(S^1) \subset S^1 \times \mathbb{C}, \quad \tau(S^1) = \{(z, v) \in S^1 \times \mathbb{C} : z \perp v\},$$

we can put $v(z) = iz \in \tau_z(S^1) \subset \mathbb{C}$. Since S^1 is an abelian group, the vector field X is invariant with respect to both left and right translations of the group $N_o \times S^1$. Moreover, K^φ is a normal subgroup of $N_o \times S^1$. By passing to the quotient spaces, we obtain an invariant (with respect to left and right translations) vector field $\widetilde{X} : W(H) \to \tau(W(H))$ such that the following diagram commutes:

$$\begin{tikzcd}
\tau(N_o \times S^1) \arrow[r, "\tau p"] & \tau(W(H)) \\
N_o \times S^1 \arrow[u, "X"'] \arrow[r, "p"] & W(H) \arrow[u, "\widetilde{X}"']
\end{tikzcd}$$

where $p : N_o \times S^1 \to N_o \times S^1/H = W(H)$ is the natural projection. □

Corollary 4.2.4. *Let Γ be a compact Lie group and $G = \Gamma \times S^1$. Then, $\Phi_1^t(G) \subset \Phi_1^+(G)$.*

4.2.2 $A_0(\Gamma)$-Module $A_1^t(\Gamma \times S^1)$ Structure

Proposition 4.2.5. *The $\mathbb{Z}$-module $A_1(G)$ admits a natural structure of an $A_0(\Gamma)$-module, where $A_0(\Gamma)$ denotes the Burnside ring, and the $A_0(\Gamma)$ -multiplication on the generators $(R) \in A_0(\Gamma)$ and $(K^{\varphi,l}) \in A_1(\Gamma \times S^1)$, is defined by the formula*

$$(R) \circ (K^{\varphi,l}) = \sum_{(L)} n_L \cdot (L^{\varphi,l}),$$

where the numbers n_L are computed using the recurrence formula (cf. [12, 114])

$$n_L = \frac{\left[n(L,R)|W(R)|n(L^{\varphi,l},K^{\varphi,l})|W(K^{\varphi,l})/S^1| - \sum_{(\tilde{L})>(L)} n(L^{\varphi,l},\tilde{L}^{\varphi,l}) n_{\tilde{L}} |W(\tilde{L}^{\varphi,l})/S^1| \right]}{|W(L^{\varphi,l})/S^1|} \tag{4.6}$$

where $n(L,R)$ and $n(L^{\varphi,l},\widetilde{L}^{\varphi,l})$ are defined by (2.10), and $|Y|$ stands for the cardinality of Y.

The following *multiplicativity property* of the primary degree plays an important role in practical computations of the primary degree (cf. [17, 114]):

Proposition 4.2.6. *Assume that (f_1, Ω_1) is an admissible pair in $\mathbb{R} \oplus V$, W is an orthogonal representation of Γ, Ω_0 is an open Γ-invariant subset of W and $f_0 : W \to W$ an Ω_0-admissible Γ-equivariant map. Then, we have*

(P7) (Multiplicativity) *The product map $f_1 \times f_0 : \mathbb{R} \oplus V \oplus W \to V \oplus W$ is $\Omega_1 \times \Omega_0$-admissible, and*

$$G\text{-Deg}\,(f_1 \times f_0, \Omega_1 \times \Omega_0) = \Gamma\text{-Deg}(f_0, \Omega_0) \circ G\text{-Deg}\,(f_1, \Omega_1),$$

where Γ-Deg$(f_0, \Omega_0) \in A_0(\Gamma)$ is the primary equivariant degree without free parameters and '$\circ$' stands for the $A_0(\Gamma)$-module multiplication provided by Proposition 4.2.5.

Examples of $A_0(\Gamma)$-module multiplication tables are listed in Appendix A3, where $\Gamma = Q_8, D_3, D_4, D_5, D_6, A_4, S_4, A_5$ and $O(2)$.

4.2.3 Twisted Primary Degree

Let Γ be a compact Lie group and $G = \Gamma \times S^1$ and $P_t : A_1^+(G) \to A_1^t(G)$ be the natural projection onto $A_1^t(G)$. Suppose that V an orthogonal G-representation, $\Omega \subset \mathbb{R} \oplus V$ an open bounded invariant subset and $f : \mathbb{R} \oplus V \to V$ an Ω-admissible G-equivariant map. Define the *twisted primary degree* (or simply *twisted degree*) of the map f on Ω by the formula

$$G\text{-Deg}^{t}(f, \Omega) := P_t(G\text{-Deg}\,(f, \Omega)). \tag{4.7}$$

Proposition 4.2.7. *Let Γ be a compact Lie group, $G = \Gamma \times S^1$, V an orthogonal G-representation, $\Omega \subset \mathbb{R} \oplus V$ an open G-invariant bounded set and $f : \mathbb{R} \oplus V \to V$ an Ω-admissible G-equivariant map. Then, the twisted primary degree defined by (4.7) satisfies the following properties:*

(P1)^t (Existence) *If* $G\text{-Deg}^t(f,\Omega) = \sum_{(H)} n_H(H)$ *is such that* $n_{H_o} \neq 0$ *for some* $(H_o) \in \Phi_1^t(G)$, *then there exists* $x \in \Omega$ *with* $f(x) = 0$ *and* $G_x \supset H_o$.

(P2)^t (Additivity) *Assume that* Ω_1 *and* Ω_2 *are two* G*-invariant open disjoint subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. *Then*

$$G\text{-Deg}^t(f,\Omega) = G\text{-Deg}^t(f,\Omega_1) + G\text{-Deg}^t(f,\Omega_2).$$

(P3)^t (Homotopy) *Suppose* $h : [0,1] \times \mathbb{R} \oplus V \to V$ *is an* Ω*-admissible* G*-equivariant homotopy. Then,*

$$G\text{-Deg}^t(h_\tau,\Omega) = \text{const}$$

(here $h_\tau := h(\tau,\cdot,\cdot)$, $\tau \in [0,1]$*).*

(P4)^t (Suspension) *Suppose that* W *is another orthogonal* G*-representation and let* U *be an open bounded* G*-invariant neighborhood of* 0 *in* W. *Then,*

$$G\text{-Deg}^t(f \times \text{Id}, \Omega \times U) = G\text{-Deg}^t(f,\Omega).$$

(P5)^t (Normalization) *Suppose* f *is a tubular map around* $G(x_o)$, $H := G_{x_o}$, $(H) \in \Phi_1^t(G)$, *with the local index* n_{x_o} *of* f *at* x_o *in a tubular neighborhood* $U_{G(x_o)}$. *Then,*

$$G\text{-Deg}^t(f, U_{G(x_o)}) = n_{x_o}(H).$$

(P6)^t (Elimination) *Suppose* f *is normal in* Ω *and* $\Omega_H \cap f^{-1}(0) = \emptyset$ *for every* $(H) \in \Phi_1^t(G,V)$. *Then,*

$$G\text{-Deg}^t(f,\Omega) = 0.$$

4.2.4 Basic Degrees with One Parameter

Definition 4.2.8. Let $\mathcal{V}_{j,l}$ be an irreducible representation of $G = \Gamma \times S^1$, $b : \mathbb{R} \oplus \mathcal{V}_{j,l} \to \mathcal{V}_{j,l}$ be the basic map associated to $\mathcal{V}_{j,l}$ and $\Omega_{j,l}$ as provided by Definition 3.3.2. Then, the twisted primary degree $\deg_{\mathcal{V}_{j,l}} := G\text{-Deg}^t(b, \mathcal{O}_j, l)$ is called the *twisted basic degree* of $\mathcal{V}_{j,l}$.

Remark 4.2.9. (i) The twisted basic degrees can be computed using the recurrence formula (cf. Proposition 3.5.3). Suppose that

$$\deg_{\mathcal{V}_{j,l}} = \sum_{(L) \in \Phi_1^t(G)} n_L(L).$$

Then,

$$n_L = \frac{\frac{1}{2}\dim \mathcal{V}_{j,l}^{H} - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L}) \cdot n_{\widetilde{L}} \cdot |W(\widetilde{L})/S^1|}{|W(L)/S^1|}. \tag{4.8}$$

(ii) As examples, the twisted basic degrees for $\Gamma = Q_8$, D_N, A_4, S_4, A_5, $O(2)$ are provided in Appendix A2.

5

Euler Ring and Equivariant Degree for Gradient Maps

One of the most important feature of the Brouwer degree is the multiplicativity property taken in the integer ring $\mathbb{Z}$. Possible extensions of this property to the primary equivariant degree are usually connected to the Burnside ring and relevant module structures (cf. Chapter 4). It turns out that the multiplicativity property is naturally valid for the so-called *equivariant degree for gradient G-maps.* This equivariant degree was introduced by K. Gęba, in order to develop equivariant degree methods for applications to the variational problems (cf. [71, 101, 153]). The gradient G-degree takes values in the so-called *Euler ring* $U(G)$, which is a generalization of the Burnside ring, introduced by T. tom Dieck in [47]. The multiplicative structure of $U(G)$ is naturally related to the multiplicativity property of the gradient equivariant degree, and is essential for its effective usage.

Therefore, a better understanding of the ring structure of $U(G)$ is essential for establishing the exact multiplication tables for several important groups. It turns out that, in the case $G = \varGamma \times S^1$, the ring structure on $U(G)$ is closely related to the previously considered algebraic structures such as the Burnside ring and $A_0(\varGamma)$-module $A_1^t(G)$ (cf. Remark 5.1.13). However, the multiplicative structure in $U(G)$, as defined in terms of Euler characteristics taken in the Alexander-Spanier cohomology with compact supports, is in general difficult to compute. Nevertheless, there are several techniques available towards this direction: (i) induction over orbit types and reasonable recurrence formulae, (ii) ring homomorphisms to other known structures $U(G_o)$ (for example taking G_o to be a maximal torus in G) (iii) fibre bundles of specific orbit spaces and techniques for computations of Euler characteristics. It is our belief that natural module structures, related to multi-parameter primary degrees, may also provide a clue to understand the algebraic structure of $U(G)$.

In the case G is a one-dimensional bi-orientable compact Lie group, we propose a passage from the gradient equivariant degree to the primary degree with one parameter, by defining the so-called *equivariant orthogonal degree* (cf. [152] for $G = \varGamma \times S^1$ with $\varGamma$ being finite), which reduces the computations of the gradient degrees to those of the primary degree and thus makes all the computational tools (related to the primary degree) available for the application of the gradient degree to variational problems. This technique is further developed in Subsection 5.2.4, where it is applied (on the H-fixed point spaces) to establish a connection between the gradient degree and twisted primary degree for the case $G = \Gamma \times S^1$ with Γ being a compact Lie group. Observe that in the case of the gradient degree, the notion of basic maps simply coincides with the map $-\mathrm{Id} : \mathcal{V} \to \mathcal{V}$, where $\mathcal{V}$ is an irreducible G-representation. We will call the corresponding gradient degrees, the *basic gradient degrees*. For convenience, the basic gradient degrees for $G = \Gamma \times S^1$ are listed in Appendix A2 for $\Gamma = Q_8, D_N, A_4, S_4, A_5$ and $O(2)$.

5.1 Euler Ring and Related Modules

5.1.1 Relation between Euler Ring, Burnside Ring and Other Related Modules

Recall the definition of the Euler ring, which was introduced in [47].

Definition 5.1.1. Let G be a compact Lie group. Consider the free $\mathbb{Z}$-module generated by $\Phi(G)$, i.e.

$$U(G) := \mathbb{Z}[\Phi(G)].$$

Define a ring multiplication $* : U(G) \times U(G) \to U(G)$, on generators (H), $(K) \in \Phi(G)$ by

$$(H) * (K) = \sum_{(L) \in \Phi(G)} n_L(L), \tag{5.1}$$

where the coefficients are given by

$$n_L := \chi_c((G/H \times G/K)_L/N(L)), \tag{5.2}$$

where χ_c stands for the Euler characteristic taken in Alexander-Spanier cohomology with compact support (cf. Section 2.6). The $\mathbb{Z}$-module $U(G)$ equipped with the multiplication $*$ is called the *Euler ring* of the group G.

Proposition 5.1.2. (GENERAL RECURRENCE FORMULA) *Given (H), $(K) \in \Phi(G)$, one has the following recurrence formula for the computations of coefficients n_L in (5.1),*

$$n_L = \chi((G/H \times G/K)^L/N(L)) - \sum_{(\widetilde{L})>(L)} n_{\widetilde{L}}\, \chi((G/\widetilde{L})^L/N(L)). \tag{5.3}$$

Proof: Let $X := G/H \times G/K$. The projection $X_{(\widetilde{L})} \to X_{(\widetilde{L})}/G$ is a fibre bundle with fibre $G/\widetilde{L}$, which implies that $X^L_{(\widetilde{L})}/N(L) \to X_{(\widetilde{L})}/G$ is a fibre bundle with fibre $((G/\widetilde{L})^L)/N(L)$. By Lemma 2.6.11, we have

$$\chi_c(X^L_{(\widetilde{L})}/N(L)) = \chi((G/\widetilde{L})^L/N(L)) \cdot \chi_c(X_{(\widetilde{L})}/G).$$

Therefore,

$$\begin{aligned}
\chi(X^L/N(L)) &= \sum_{(\widetilde{L})\geq(L)} \chi_c(X^L_{(\widetilde{L})}/N(L)) \\
&= \sum_{(\widetilde{L})\geq(L)} \chi((G/\widetilde{L})^L/N(L)) \cdot \chi_c(X_{(\widetilde{L})}/G) \\
&= \sum_{(\widetilde{L})\geq(L)} \chi((G/\widetilde{L})^L/N(L)) \cdot \chi_c(X_{\widetilde{L}}/N(\widetilde{L})) \\
&= \sum_{(\widetilde{L})\geq(L)} \chi((G/\widetilde{L})^L/N(L)) \cdot n_{\widetilde{L}} \\
&= n_L + \sum_{(\widetilde{L})>(L)} n_{\widetilde{L}} \cdot \chi((G/\widetilde{L})^L/N(L))
\end{aligned}$$

and the result follows.

□

The following fact plays an essential role in our computations of the multiplication structure in $U(G)$.

Proposition 5.1.3. *Let H, $\tilde{H}$ be subgroups of G such that* $\dim W(H) = \dim W(\widetilde{H}) = 1$. *Assume that for any maximal orbit type (L_o) in the G-space $G/H \times G/\widetilde{H}$, the group L_o is finite. Let*

$$(H) * (\widetilde{H}) = \sum_{(L)\in\Phi(G)} n_L(L). \tag{5.4}$$

Then, $n_L = 0$ for any finite subgroup $L \subset G$ with $\dim W(L) = 1$.

Proof: Take a finite subgroup $L \subset G$ with $\dim W(L) = 1$. Clearly, $\dim N(L) = 1$. Consider $(G/H)^L$ as a left $N(L)$-space. By Proposition 2.4.3,

$(G/H)^L$ is diffeomorphic to the right $N(L)$-space $N(L,H)/H$. By the assumption that $\dim W(H) = \dim W(L) = 1$, $N(L,H)/H$ is a closed 1-dimensional submanifold of G/H (cf. Proposition 2.4.5(iii)). Similarly, $N(L,\widetilde{H})/\widetilde{H}$ is also a compact 1-dimensional manifold. Put

$$X := \left(G/H \times G/\widetilde{H}\right)^L = (G/H)^L \times (G/\widetilde{H})^L,$$

which is then diffeomorphic to a compact 2-dimensional manifold $N(L,H)/H \times N(L,\widetilde{H})/\widetilde{H}$.

We claim that each connected component of X has one orbit type (in fact, one isotropy) under the $N(L)$-action. By a connected component of X, we mean the product space of two S^1-orbits in $N(L,H)/H$ and $N(L,\widetilde{H})/\widetilde{H}$ respectively (where $S^1 \subset N(L)$ is the connected component of $e \in N(L)$), namely $S^1(Hg) \times S^1(\widetilde{H}\widetilde{g})$ for some $g, \widetilde{g} \in G$. Notice that when S^1 moves $(Hg, \widetilde{H}\widetilde{g})$ to $(Hg\gamma, \widetilde{H}\widetilde{g}\widetilde{\gamma})$ for some $\gamma, \widetilde{\gamma} \in S^1$, the corresponding isotropy changes from $g^{-1}Hg \cap \widetilde{g}^{-1}\widetilde{H}\widetilde{g}$ to $\gamma^{-1}(g^{-1}Hg)\gamma \cap \widetilde{\gamma}^{-1}(\widetilde{g}^{-1}\widetilde{H}\widetilde{g})\widetilde{\gamma}$. It suffices to show that $\gamma^{-1}(g^{-1}Hg)\gamma = g^{-1}Hg$ and $\widetilde{\gamma}^{-1}(\widetilde{g}^{-1}\widetilde{H}\widetilde{g})\widetilde{\gamma} = \widetilde{g}^{-1}\widetilde{H}\widetilde{g}$. We only prove the first equality (for arbitrary $\gamma \in S^1$), which is equivalent to show that $S^1 \subset N(g^{-1}Hg)$. By assumption $\dim W(H) = 1$, we have that $\dim N(g^{-1}Hg) = \dim N(H) \geq \dim W(H) = 1$, which certainly implies that $N(g^{-1}Hg)$ contains S^1.

Consequently, the right $N(L)$-space X, though may have different orbit types, each of its connected component shares the same orbit type. Since each connected component is both open and closed, the structure theorem, though initially designed for homogeneous spaces, remains valid, which claims that $X/N(L)$ is a smooth manifold. To determine the dimension, it is enough to notice that, by assumption, $N(L)$ acts on X by finite isotropies, hence $X/N(L)$ is a compact smooth manifold of dimension 1. Thus, $\chi(X/N(L)) = 0$.

In the case (L) is a maximal type in $G/H \times G/\widetilde{H}$, then

$$X = \left(G/H \times G/\widetilde{H}\right)_L.$$

Hence, $n_L = \chi(X/N(L)) = 0$.

In the case (L) is not a maximal orbit type in $G/H \times G/\widetilde{H}$, then

$$\begin{aligned} n_L &= \chi_c\left(\left(G/H \times G/\widetilde{H}\right)_L /N(L)\right) \\ &= \chi(X/N(L)) - \sum_{(L')>(L)} \chi_c(\left(G/H \times G/\widetilde{H}\right)_{L'} /N(L'). \end{aligned}$$

By induction on the lattice of orbit types in $G/H \times G/\widetilde{H}$, we obtain that

$$\chi_c\left(\left(G/H \times G/\widetilde{H}\right)_{L'} /N(L')\right) = 0,$$

for all finite $L' \subset G$ and $\dim W(L') = 1$. □

Example 5.1.4. Let $G := O(2) \times S^1$. Then, we have that (we refer to Appendix A2 for conventions)

$$\begin{aligned} \Phi_0(G) &= \{(O(2) \times S^1), (SO(2) \times S^1), (D_n \times S^1)\}, \\ \Phi_1(G) &= \{(\mathbb{Z}_n \times S^1), (O(2) \times \mathbb{Z}_l), (SO(2) \times \mathbb{Z}_l), (D_n \times \mathbb{Z}_l), \\ &\qquad (O(2)^{-,l}), (SO(2)^{\varphi_k,l}), (D_n^{z,l}), (D_{2n}^{d,l})\} \\ \Phi_2(G) &= \{(\mathbb{Z}_n \times \mathbb{Z}_l), (\mathbb{Z}_n^{\varphi_k,l}), (\mathbb{Z}_{2n}^{d,l})\} \end{aligned}$$

(a) Take $H = D_n \times \mathbb{Z}_l$, $\tilde{H} = \mathbb{Z}_m \times S^1$. Notice that $(H), (\tilde{H}) \in \Phi_1(G)$, i.e. $\dim W(H) = \dim W(\tilde{H}) = 1$. Moreover, any isotropy subgroup in the G-space $G/H \times G/\tilde{H}$ has the form of $g_1 H g_1^{-1} \subset g_2 \tilde{H} g_2^{-1}$, for some $g_1, g_2 \in G$. Since H is finite, we have that this isotropy must be finite as well. Therefore, by Proposition 5.1.3, we have that $n_L = 0$ in (5.4) for $(L) \in \{(D_n \times \mathbb{Z}_l), (D_n^{z,l}), (D_{2n}^{d,l})\}$.

(b) Using the argument similar to the one used in the proof of Proposition 5.1.3, one can show that if H and K are subgroups of G with $\dim W(H) \geq 1$ and $\dim W(K) = 2$. Then,

$$(H) * (K) = 0.$$

Indeed, assume that for some $(L) \in \Phi(G)$ one has that the coefficient n_L in $(H) * (K)$ is different from [illegible]. Then, $(L) \leq (K)$ which, by assumption and Proposition 2.4.5(i), impl[illegible] $W(L) = 2$. In particular,

$$N(L) \supset SO(2) \times S^1 = T^2. \tag{5.5}$$

Consider the space

$$X := (G/H \times G/K)^L = (G/H)^L \times (G/K)^L = N(L, K)/K \times N(L, H)/H.$$

Combining (5.5) with Proposition 2.4.11 implies that $N(L,H)$ and $N(L,K)$ contain T^2. Therefore, $N(L,H)/H$ and $N(L,K)/K$ admit T^2-actions without T^2-fixed-points. By Lemma 2.6.13, $\chi(X/T^2) = 0$. If $N(L) = T^2$, then $\chi(X/N(L)) = 0$. Another possibility for $N(L)$ may be $N(L) = O(2) \times S^1$. Then, using the same fibre bundle argument as in the proof of Proposition 5.1.3 one concludes that $\chi(X/N(L)) = 0$ as well. If (L) is a maximal orbit type in X, then the last equality implies $n_L = 0$. If (L) is not maximal, one can use the same induction argument as in the proof of Proposition 5.1.3 to show that $n_L = 0$.

Burnside Ring

Recall that the Burnside ring $A_0(G)$ is defined as the $\mathbb{Z}$-module $A_0(G) := \mathbb{Z}[\Phi_0(G)]$ equipped with a similar multiplication as in $U(G)$ but restricted only to regenerators from $\Phi_0(G)$ (cf. Section 4.1.1), i.e. for (H), $(K) \in \Phi_0(G)$

$$(H) \cdot (K) = \sum_{(L)} n_L(L) \qquad ((H),(K),(L) \in \Phi_0(G)),$$

where $n_L := \chi((G/H \times G/K)_L/N(L)) = |(G/H \times G/K)_L/N(L)|$ (here χ stands for the usual Euler characteristic). One can easily notice that the space $(G/H \times G/K)_{(L)}/G$ is finite, thus

$$\chi((G/H \times G/K)_{(L)}/G) = |(G/H \times G/K)_{(L)}/G|,$$

where $|X|$ stands for the number of elements in X.

Observe that being a $\mathbb{Z}$-submodule of $U(G)$, the Burnside ring $A_0(G)$ may *not* be a subring of $U(G)$, in general. Indeed, we have the following example

Example 5.1.5. Let $G = O(2)$. By direct computation, we have $(D_n) \cdot (SO(2)) = 0$, while $(D_n) * (SO(2)) = (\mathbb{Z}_n)$.

However, there is a connection between the rings $U(G)$ and $A_0(G)$. Take the natural projection $\pi_0 : U(G) \to A_0(G)$ defined on generators $(H) \in \Phi(G)$ by

$$\pi_0((H)) = \begin{cases} (H) & \text{if } (H) \in \Phi_0(G), \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Lemma 5.1.6. *The map π_0 defined by (5.6) is a ring homomorphism, i.e.*

$$\pi_0((H) * (K)) = \pi_0((H)) \cdot \pi_0((K)), \quad (H), (K) \in \Phi(G).$$

Proof: Assume $(H) \notin \Phi_0(G)$ and

$$(H) * (K) = \sum_{(R) \in \Phi(G)} m_R(R) \qquad ((R) \in \Phi(G)). \tag{5.7}$$

Then, for any (R) occuring in (5.7), one has $(R) \leq (H)$, hence $\dim W(R) > 0$ (cf. Proposition 2.4.5(i)). By definition of π, $\pi_0((R)) = 0$ and thus $\pi_0((H) * (K)) = 0$. On the other hand, $\pi_0((H)) \cdot \pi_0((K)) = 0 \cdot \pi_0(K) = 0$.

Thus, without loss of generality, assume $(H), (K) \in \Phi_0(G)$ and

$$(H) * (K) = \sum_{(L) \in \Phi_0(G)} n_L(L) + \sum_{(\tilde{L}) \notin \Phi_0(G)} m_{\tilde{L}}(\tilde{L}).$$

Then,

$$\pi_0((H) * (K)) = \sum_{(L) \in \Phi_0(G)} n_L \pi_0((L)) = \sum_{(L) \in \Phi_0(G)} n_L(L).$$

and

$$(H) \cdot (K) = \sum_{(L) \in \Phi_0(G)} n'_L(L).$$

However,

$$\begin{aligned} n_L &= \chi_c((G/H \times G/K))_L)/N(L)) \\ &= \chi((G/H \times G/K))_L/N(L)) \\ &= |(G/H \times G/K)_L/N(L)| \\ &= n'_L \end{aligned} \tag{5.8}$$

and the result follows. □

The following stated result is due to T. tom Dieck (cf. [47]). We provide an alternative proof.

Proposition 5.1.7. *Let $(H) \in \Phi_n(G)$ with $n > 0$. Then, (H) is a nilpotent element in $U(G)$, i.e. there is an integer k such that $(H)^k = 0$ in $U(G)$.*

Proof: We will use induction and the fact that there is only finitely many conjugacy classes of isotropies in the spaces $\underbrace{G/H \times \dots \times G/H}_{k \text{ times}}$. Suppose that for $k \geq 1$ we have the expansion

$$(H)^k = \sum_{(K)} a_K(K), \tag{5.9}$$

and assume that (L) a maximal element in the sum (5.9) with $a_L \neq 0$. We will show that the expansion of the product $(H)^{k+1}$ does not contain the term (L) with non-zero coefficient. Indeed, by multiplying (5.9) by (H) we obtain

$$(H)^{k+1} = \sum_{(K)} a_K(K) * (H), \tag{5.10}$$

then by the maximality of (L) we obtain that the only product $(K) * (H)$ in (5.10) that can lead to a term with (L)-coefficient is $(L) * (H)$. Notice that (L) is the maximal orbit type in $G/H \times G/L$, thus

$$(G/H \times G/L)_L = (G/H \times G/L)^L = (G/H)^L \times (G/L)^L = (G/H)^L \times N(L)/L.$$

Notice that (see Corollary 1.92 in [104])

$$((G/H)^L \times N(L)/L)/N(L) = (G/H)^L \times W(L))/W(L) = (G/H)^L.$$

Hence

$$\chi_c((G/H \times G/L)_L/N(L)) = \chi(((G/H)^L \times N(L)/L)/N(L)) = \chi(((G/H)^L).$$

Since $W(H)$ acts freely on $(G/H)^L = N(L,H)/H$ and $\dim W(H) > 0$, the maximal torus $T^m \subset W(H)$ (with $m \geq 1$) acts freely on $(G/H)^L$, which means $((G/H)^L)^{T^m} = \emptyset$. Then by Proposition 2.6.12, $\chi((G/H)^L) = 0$, and the conclusion follows. ☐

Combining Proposition 5.1.7 with Lemma 5.1.6 and the fact that the multiplication table for $A_0(G)$ contains only non-negative coefficients (cf. formula (5.8)), yields

Proposition 5.1.8. (cf. [73]) *Let π_0 be defined by (5.6). Then, $\mathfrak{N} = \ker \pi_0 = \mathbb{Z}[\Phi(G) \setminus \Phi_0(G)]$ is a maximal nilpotent ideal in $U(G)$ and $A_0(G) \cong U(G)/\mathfrak{N}$.*

Summing up, the Burnside ring multiplication structure in $A_0(G)$ can be used to describe (partially) the Euler ring multiplication structure in $U(G)$.

Twisted Subgroups and Related Modules

We resume the assumption that $G = \Gamma \times S^1$, where Γ is a compact Lie group. In this case, there are exactly two sorts of subgroups $H \subset G$, namely,

(a) $H = K \times S^1$, for $K \subset \Gamma$;
(b) φ-twisted l-folded subgroups $K^{\varphi,l}$ (cf. Subsection 4.2.1).

Proposition 5.1.9. *Let $G = \Gamma \times S^1$, where Γ is a compact Lie group. Given a twisted subgroup $K^{\varphi,l} \subset G$, for some $l \in \{0\} \cup \mathbb{N}$ and a homomorphism $\varphi : K \to S^1$, the following holds*

$$\dim\big(N_G(K^{\varphi,l})\big) = \dim\big(N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi)\big) + 1. \tag{5.11}$$

Proof: For the homomorphism $\varphi : K \to S^1$, put $L := \ker\varphi$. Also, for simplicity, write $N(K^{\varphi,l})$ for $N_G(K^{\varphi,l})$, and $N(K)$ (resp. $N(L)$) for $N_\Gamma(K)$ (resp. $N_\Gamma(L)$).

Notice that $N(K^{\varphi,l}) = N_o \times S^1$, where

$$N_o := \{\gamma \in N(K) \;:\; \varphi(\gamma k \gamma^{-1}) = \varphi(k),\ \forall k \in K\}.$$

Hence, it is sufficient to show that $\dim N_o = \dim\big(N(K) \cap N(L)\big)$.

Case 1. φ is surjective.

By the fundamental homomorphism theorem of algebra, we have $K/L \simeq S^1$. Fix an element $t \in N(K) \cap N(L)$, define an automorphism $h_\gamma : K \to K$ by $h_\gamma(k) := \gamma k \gamma^{-1}$. Since $\gamma \in N(L)$, h_γ induces a homomorphism on the factor group K/L, which will be denoted by $\bar{h}_\gamma$. Then, we have the commutative diagram shown in Figure 1.

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "h_\gamma"] & K/L \simeq S^1 \arrow[d, "\bar{h}_\gamma"] \\
K \arrow[r, "\varphi"] & K/L \simeq S^1
\end{tikzcd}$$

Fig. 5.1. Commutative diagram for surjective φ.

For any fixed γ in the connected component of $e \in N(K) \cap N(L)$, let σ_γ be a path from γ to e. Then, this path induces a homotopic homomorphism

h_{σ_γ} connecting h_γ to Id on K, as well as a homotopic homomorphism $\bar{h}_{\sigma_\gamma}$ connecting $\bar{h}_\gamma$ to Id on $K/L \simeq S^1$. It is well-known that any group automorphism on S^1 has the form of $z \mapsto z^n$ for some $n \in \mathbb{Z}$, and each represents a distinct homotopy class in $H_1(S^1;\mathbb{Z})$. Thus, we conclude that $\bar{h}_\gamma \equiv \mathrm{Id}$. By the commutative diagram in Figure 5.1.1, it is equivalent to claim that $\varphi \circ h_\gamma \equiv \varphi$, i.e. $\varphi(\gamma k \gamma^{-1}) = \varphi(k)$ for all $k \in K$. Therefore, every γ in the same connected component of $e \in N(K) \cap N(L)$, actually belongs to N_o. This implies that $\dim\big(N(K) \cap N(L)\big) \leq \dim N_o$. On the other hand, by direct verification, $N_o \subset N(K) \cap N(L)$. Therefore, $\dim N_o = \dim\big(N(K) \cap N(L)\big)$.

Case 2. φ is not surjective.

Take any element γ in the same connected component of $e \in N(K) \cap N(L)$, and denote by σ_γ a path from γ to e. Define $\varphi_\sigma : [0,1] \times K \to S^1$ by $\varphi_\sigma(t,k) := \varphi\big(\sigma_\gamma(t) k (\sigma_\gamma(t))^{-1}\big)$. Since φ is not surjective, φ_σ has a discrete image in S^1. Hence, when restricted on a connected component, φ_σ is constant, so we have $\varphi(\gamma k \gamma^{-1}) = \varphi(k)$ for all k in the same connected component of K. Therefore, for any element γ in the same connected component of $e \in N(K)$, we have $\varphi(\gamma k \gamma^{-1}) = \varphi(k)$ for all $k \in K$, i.e. $\gamma \in N_o$, which implies $\dim N(K) \cap N(L) \leq \dim N_o$. On the other hand, $N_o \subset N(K) \cap N(L)$. Therefore, $\dim N_o = \dim\big(N(K) \cap N(L)\big)$. □

Lemma 5.1.10. *Let Γ be a compact Lie group, $G = \Gamma \times S^1$ and $H = K^{\varphi,l} \subset G$ a twisted subgroup. Then,*

(i) $1 \leq \dim W_G(H) \leq 1 + \dim W_\Gamma(K)$;
(ii) any subgroup $\tilde{H} \subset H$ is twisted:

Proof: (i) The second inequality was established in [15], Section 5.1. To prove the first inequality, observe that $N_G(K^{\varphi,l}) = N_o \times S^1$ with $K \subset N_o \subset N_\Gamma(K)$. Thus,

$$W_G(K^{\varphi,l}) = \frac{N_o \times S^1}{K^{\varphi,l}} \supset \frac{K \times S^1}{K^{\varphi,l}}. \tag{5.12}$$

Consider a homomorphism $\psi : K \times S^1 \to S^1$ defined by $\psi(\gamma, z) = \varphi(\gamma) z^{-l}$. Since ψ is surjective and $\ker \psi = K^{\varphi,l}$, we obtain that $\dim K \times S^1/K^{\varphi,l} = \dim S^1 = 1$ from which (cf. (5.12)) the statement follows.

(ii) It is obvious that $\tilde{H}$ is twisted by the same homomorphism φ. □

Corollary 5.1.11. *Let G be as in Lemma 5.1.10.*

(a) Let H be a twisted subgroup of G. Then, $\dim W_G(H) = 1$ *if* $\dim W_\Gamma(K) = 0$.
(b) $\Phi_0(G) = \{(H) : H \subset G,\ H = K \times S^1,\ \dim W_\Gamma(K) = 0\}$ *and thus*

$$A_0(G) \cong A_0(\Gamma). \tag{5.13}$$

(c) If $H = K^{\varphi,l}$ *is twisted in* G, $\dim W(H) = 1$ *and* $(H) < (\widetilde{H}) \in \Phi_1(G)$, *then* $\widetilde{H}$ *is twisted in* G *and* $\dim W(\widetilde{H}) = 1$.

Proof: Statement (a) follows directly from Lemma 5.1.10(i). Next, Lemma 5.1.10(i) excludes twisted conjugacy classes from $\Phi_0(G)$. Since, for $H = K \times S^1$ for $K \subset \Gamma$, on has $\dim W_G(H) = 0$ if and only if $\dim W_\Gamma(K) = 0$. Hence, the statement (b) follows.

To prove (c), observe that $\widetilde{H}$ cannot be a subgroup of type $\widetilde{K} \times S^1$, since it would imply $\dim W_\Gamma(\widetilde{K}) = 1$ and $(K) \leq (\widetilde{K})$, which would be a contradiction to Proposition 2.4.5(i) combined with (a). Consequently, $\widetilde{H} = \widetilde{K}^{\psi,m}$, where $\psi : \widetilde{K} \to S^1$ is a homomorphism, and since $K \subset \widetilde{K}$, it follows that $\dim W_\Gamma(\widetilde{K}) = 0$, which implies that $\dim W(\widetilde{H}) = 1$ (cf. (a)).

□

Being motivated by Corollary 5.1.11, put

$$\begin{aligned}
\Phi_1^t(G) &:= \{(H) \in \Phi(G) :\ H = K^{\varphi,l} \text{ for some } K \subset \Gamma \text{ with } \dim W_\Gamma(K) = 0\},\\
\Phi_1^*(G) &:= \{(H) \in \Phi(G) :\ \dim W_G(H) = 1 \text{ and } (H) \notin \Phi_1^t(G)\},\\
\Phi_k^*(G) &:= \{(H) \in \Phi(G) :\ \dim W_G(H) = k\},\quad k \geq 2,
\end{aligned}$$

and define

$$\begin{aligned}
A_1^t(G) &:= \mathbb{Z}[\Phi_1^t(G)],\\
A_k^*(G) &:= \mathbb{Z}[\Phi_k^*(G)],\quad k \geq 1,\\
A^*(G) &:= \bigoplus_{k\geq 1} A_k^*(G).
\end{aligned}$$

As it was discussed in Subsection 4.2.2, there is a natural $A_0(\Gamma)$-module structure on $A_1^t(G)$ (cf. Proposition 4.2.5). By using Corollary 5.1.11, one can establish a relation between the $A_0(\Gamma)$-module structure on $A_1^t(G)$ and the ring structure on $U(G)$.

To this end, take the natural projection $\pi_1 : U(G) \to A_1^t(G)$ defined by

$$\pi_1(H) = \begin{cases} (H) & \text{if } (H) \in \Phi_1^t(G), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.1.12. *Let Γ be a compact Lie group and $G = \Gamma \times S^1$. If $(\widetilde{H}) \in \Phi_0(G)$ with $\widetilde{H} = K \times S^1$ and $(H) \in \Phi_1^t(G)$, then*

$$\pi_1((\widetilde{H}) * (H)) = (K) \circ (H).$$

Remark 5.1.13. Proposition 5.1.12 indicates that the multiplication table in the $\mathbb{Z}$-module decomposition $U(G) = A_0(G) \oplus A_1^t(G) \oplus A^*(G)$ can be described by the following table

$*$	$A_0(G) \cong A_0(\Gamma)$	$A_1^t(G)$	$A^*(G)$
$A_0(G) \cong A_0(\Gamma)$	$A_0(G)$-multip $+T_\bullet$	$A_0(\Gamma)$-module multip $+T_\bullet$	$T_\bullet$
$A_1^t(G)$	$A_0(\Gamma)$-module multip $+T_\bullet$	$T_1 + T_\bullet$	$T_\bullet$
$A^*(G)$	$T_\bullet$	$T_\bullet$	$T_\bullet$

where $T_\bullet$ stands for an element from $A^*(G)$ and T_1 for an element from $A_1^t(G)$.

Table 5.1. $U(G)$-Multiplication Table for $G = \Gamma \times S^1$

In the case Γ is a finite group, we have the following result (cf. [152])

Proposition 5.1.14. *For $G = \Gamma \times S^1$ with Γ being a finite group, the multiplication in $U(G)$, when restricted to $A_1(G) \times A_1(G)$, is trivial, i.e. for any $(H), (K) \in \Phi_1(G)$, we have*

$$(H) * (K) = 0.$$

Proof: Let (H), $(K) \in \Phi_1(G)$. Take $L \subset G$ such that $(G/H \times G/K)^L \neq \emptyset$. By dimension restrictions, we have $(L) \in \Phi_1(G)$ (cf. Proposition 2.4.5(i)).

Claim. $\chi\Big((G/H \times G/K)^L/W(L)\Big) = 0$ for $(L) \in \Phi_1(G)$.

We prove the claim by showing that $(G/H \times G/K)^L/W(L)$ allows an S^1-action without S^1-fixed points.

Observe that $(G/H \times G/K)^L = (G/H)^L \times (G/K)^L$. By Proposition 2.4.3, the space $(G/H)^L$ is homeomorphic to $N(L,H)/H$, on which $W(H)$ acts freely. Thus, the space $(G/H)^L$ is of dimension 1. On the other hand, by Proposition 2.4.4, $(G/H)^L$ is composed of a finite number of $W(L)$-orbits. Therefore, by the dimension restriction, the isotropy subgroup $W(L)_x$ of each point $x \in (G/H)^L$ is finite.

Take the connected component of the neutral element $e \in W(L)$, which is diffeomorphic to S^1. Consider the $W(L)$-space $(G/H)^L$ as an S^1-space. For each $x \in (G/H)^L$, the new isotropy is $S^1_x = W(L)_x \cap S^1$, which forces S^1_x to be finite. Consequently, every connected component of $(G/H)^L$ allows an S^1-action without S^1-fixed points.

Similarly, every connected component of $(G/K)^L$ allows an S^1-action without S^1-fixed points. Consider the product space $(G/H)^L \times (G/K)^L$ as an S^1-space by the diagonal action. Then, by Lemma 2.6.13, we have

$$\chi\big(((G/H)^L \times (G/K)^L)/S^1\big) = 0.$$

To conclude that $\chi\big(((G/H)^L \times (G/K)^L)/W(L)\big) = 0$, it is sufficient to observe that $((G/H)^L \times (G/K)^L)/S^1 \to ((G/H)^L \times (G/K)^L)/W(L)$ is a trivial fibre bundle with a finite fibre $W(L)/S^1$. $\square_{Claim}$

If (L) is a maximal orbit type in $(G/H)^L \times (G/K)^L$, then

$$\begin{aligned} n_L &= \chi_c\big((G/H \times G/K)_L/W(L)\big) \\ &= \chi\big((G/H \times G/K)^L/W(L)\big) \\ &= 0. \end{aligned}$$

Otherwise, one applies the general recurrence formula (cf. Proposition 5.1.2) and conclude that $n_L = 0$.

$\square$

In the rest of this subsection, we present the computational formulae for the Euler ring $U(T^n)$, where T^n is an n-dimensional torus. The following statement was observed by S. Rybicki.

Proposition 5.1.15. *If* (H), $(K) \in \Phi(T^n)$, *and* $L = H \cap K$, *then*

$$(H) * (K) = \begin{cases} (L) & \text{if } \dim H + \dim K - \dim L = \dim T^n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Put $G := T^n$ and observe that every compact abelian connected Lie group is a torus. Since H and K are normal in G, the groups G/H and G/K are tori. Take $L = H \cap K$. Since G is abelian, L is the only one isotropy in $(G/H \times G/K)^L$ with respect to the $N(L) = G$-action. Hence,

$$(H) * (K) = \chi\left((G/H \times G/K)^L \big/ G\right)(L)$$

Next, $N(L, H) = G$, therefore

$$(G/H \times G/K)^L \big/ G = (G/H \times G/K) \big/ G.$$

Put $M := (G/H \times G/K) \big/ G$. Observe that M is a compact connected G-manifold of precisely one orbit type (L). Thus, it is of dimension $N := \dim G/H + \dim G/K - \dim G + \dim L = \dim G - \dim K - \dim H + \dim L$. If $N := 0$, then $\chi(M) = 1$, and if $N > 0$, then there is an action of a torus on M without G-fixed-points, so $\chi(M) = 0$ (cf. Lemma 2.6.12). □

The full multiplication table for $U(T^2)$ is presented in A3.19, Appendix A3.

5.1.2 Euler Ring Homomorphism

Let $\psi : G' \to G$ be a homomorphism between compact Lie groups. Then, the formula $g'x := \psi(g')x$ defines a left G'-action on G. In particular, for any subgroup $H \subset G$, the map ψ induces the G'-action on G/H with

$$G'_{gH} = \psi^{-1}(gHg^{-1}). \tag{5.14}$$

In this way, ψ induces a map $\Psi : U(G) \to U(G')$ defined by

$$\Psi((H)) := \sum_{(H')\in\Phi(G')} \chi_c((G/H)_{(H')}/G')(H'). \tag{5.15}$$

We claim that

Lemma 5.1.16. *The map Ψ defined by (5.15) is the Euler ring homomorphism.*

Proof: Recall that, by Gleason Lemma, if X is a compact G-CW complex, then the projection map $X_{(H)} \to X_{(H)}/G$ is a fibre bundle with the fibre G/H (cf. [25], p. 88, Theorem 5.8). Hence, by Lemma 2.6.11,

$$\chi_c(X) = \sum_{(H)} \chi_c(X_{(H)}), \quad \chi_c(X_{(H)}) = \chi_c(X_{(H)}/G) \cdot \chi(G/H).$$

Combining the formulae (5.2), (5.15), Lemma 2.6.11, one obtains

$$\begin{aligned}
\Psi((H) * (K)) &= \Psi(\sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G) \cdot (L)) \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G) \cdot \Psi(L) \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G) \sum_{(L')} \chi_c((G/L)_{(L')}/G') \cdot (L') \\
&= \sum_{(L')} \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G) \chi_c((G/L)_{(L')}/G') \cdot (L').
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Psi(H) * \Psi(K) \\
&= \sum_{(H')} \chi_c((G/H)_{(H')}/G') \cdot (H') * \sum_{(K')} \chi_c((G/K)_{(K')}/G') \cdot (K') \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G') \chi_c((G/K)_{(K')}/G') \cdot (H') * (K') \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G') \chi_c((G/K)_{(K')}/G') \cdot \sum_{(L')} \chi_c((G'/H' \times G'/K')_{(L')}/G') \cdot (L') \\
&= \sum_{(L')} \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G') \chi_c((G/K)_{(K')}/G') \chi_c((G'/H' \times G'/K')_{(L')}/G') \cdot (L').
\end{aligned}$$

Put

$$\begin{aligned}
n_{L'} &:= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G) \chi_c((G/L)_{(L')}/G'), \\
m_{L'} &:= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G') \chi_c((G/K)_{(K')}/G') \chi_c((G'/H' \times G'/K')_{(L')}/G').
\end{aligned}$$

We need to show that for all G'-orbit types (L') in $G/H \times G/K$

$$n_{L'} = m_{L'}. \tag{5.16}$$

Consider $u_{L'} := \chi_c((G/H \times G/K)_{(L')}/G') = \chi_c((G/H \times G/K)_{L'}/N(L'))$. If (L') is a maximal orbit type, then

$$u_{L'} = \chi_c(G/H \times G/K)_{L'}/N(L') = \chi_c(G/H \times G/K)^{L'}/N(L') = \sum_{(L)} \chi_c(G/H \times G/K)^{L'}_{(L)}/N(L'),$$

where the union is taken over all (L)-orbit types occuring in $(G/H \times G/K)^{L'}$ (considered as $N(\psi(L'))$-space) (cf. (5.14)). Using the fibre bundle $G/L \hookrightarrow (G/H \times G/K)_{(L)} \to (G/H \times G/K)_{(L)}/G$, we get that $(G/H \times$

$G/K)^{L'}_{(L)}/N(L') \to (G/H \times G/K)_{(L)}/G$ is a fibre bundle with the fibre $(G/L^{L'})/N(L')$. Thus,

$$\begin{aligned}
u_{L'} &= \chi((G/H \times G/K)^{L'}/N(L')) = \sum_{(L)} \chi_c((G/H \times G/K)^{L'}_{(L)}/N(L')) \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi((G/L^{L'})/N(L')) \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi_c(((G/L)_{L'})/N(L')) = n_{L'}
\end{aligned}$$

In the case (L') is not a maximal orbit type, assume, by induction, that $u_{\tilde{L}'} = n_{\tilde{L}'}$ for all $(\tilde{L}') > (L')$. Then,

$$\begin{aligned}
u_{L'} &= \chi_c((G/H \times G/K)_{L'}/N(L')) \\
&= \chi((G/H \times G/K)^{L'}/N(L')) - \sum_{(\tilde{L}')>(L')} \chi_c((G/H \times G/K)_{\tilde{L}'}/N\tilde{L}') \\
&= \chi((G/H \times G/K)^{L'}/N(L')) - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi((G/L^{L'})/N(L')) - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(\tilde{L}')\geq(L')} \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi((G/L_{\tilde{L}'})/N\tilde{L}') - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(\tilde{L}')\geq(L')} n_{\tilde{L}'} - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} = n_{L'} + \sum_{(\tilde{L}')>(L')} (n_{\tilde{L}'} - u_{\tilde{L}'}) = n_{L'}
\end{aligned}$$

On the other hand, in the case (L') is a maximal orbit type,

$$(G/H \times G/K)_{L'}/N(L') = (G/H \times G/K)^{L'}/N(L') = \bigcup_{(H'),(K')} ((G/H)_{(H')} \times (G/K)_{(K')})^{L'}/N(L'),$$

where the union is taken over all (H')-orbit types (resp. (K')-orbit types) occuring in $(G/H)^{L'}$ (resp. in $(G/K)^{L'}$), considered as $N(L')$-space. By using the fibre bundles $G'/H' \hookrightarrow (G/H)_{(H')} \to (G/H)_{(H')}/G'$ and $G'/K' \hookrightarrow (G/K)_{(K')} \to (G/K)_{(K')}/G'$ we obtain the product bundle $G'/H' \times G'/K' \hookrightarrow (G/H)_{(H')} \times (G/K)_{(K')} \to (G/H)_{(H')}/G' \times (G/K)_{(K')}/G'$. Therefore,

$$((G/H)_{(H')} \times (G/K)_{(K')})^{L'}/N(L') \to (G/H)_{(H')}/G' \times (G/K)_{(K')}/G'$$

is a fibre bundle with the fibre $(G'/H' \times G'/K')^{L'}/N(L')$. Consequently,

$$\begin{aligned}
u_{L'} &= \chi((G/H \times G/K)^{L'}/N(L')) = \sum_{(H'),(K')} \chi(((G/H)_{(H')} \times (G/K)_{(K')})^{L'}/N(L')) \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G' \times (G/K)_{(K')}/G')\chi((G'/H' \times G'/K')^{L'}/N(L')) \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G' \times (G/K)_{(K')}/G')\chi((G'/H' \times G'/K')_{L'}/N(L')) \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G')\chi((G'/H' \times G'/K')_{L'}/N(L')) = m_{L'}
\end{aligned}$$

In the case (L') is not a maximal orbit type, by applying induction over the orbit types in the same way as above,

$$\begin{aligned}
\chi_c((G/H \times G/K)_{L'}/N(L')) &= \chi((G/H \times G/K)^{L'}/N(L')) - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(H'),(K')} (\chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G') \\
&\qquad \cdot \chi((G'/H' \times G'/K')^{L'}/N(L'))) - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(\tilde{L}')\geq(L)} m_{\tilde{L}'} - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} = m_{L'} + \sum_{(\tilde{L}')>(L)} (m_{\tilde{L}'} - u_{\tilde{L}'}) = m_{L'}
\end{aligned}$$

Therefore, the statement follows. □

Remark 5.1.17. A similar result was obtained implicitly in [47], with a proof containing several omissions. We present hereby the proof of Lemma 5.1.16 for completeness.

5.1.3 Euler Ring Structure on $U(O(2) \times S^1)$

To establish the Euler ring multiplication on $U(O(2) \times S^1)$, we discuss the ring homomorphism $\psi : \tilde{G} \to G$ for the case $\tilde{G} = T^n$ being a maximal torus in G and $\psi : T^n \to G$ being the natural embedding. Then, the homomorphism Ψ takes the form

$$\Psi(H) = \sum_{(K)\in\Phi(T^n)} \chi_c((G/H)_{(K)}/T^n) \cdot (K), \tag{5.17}$$

with $K = H' \cap T^n$, $H' \in (H)$. Observe that since all the maximal tori in a compact Lie group are conjugate (see, for instance, [27]), the homomorphism (5.17) is independent of a choice of a maximal torus in G.

We will show that Ψ can be used to find additional coefficients for the multiplication formulae in $U(G)$. To compute Ψ, we start with the following

Proposition 5.1.18. *Let T^n be a maximal torus in G and the homomorphism Ψ is defined by (5.17). Then,*

$$\Psi(T^n) = |W(T^n)|(T^n) + \sum_{(T')} n_{T'}(T'),$$

where $T' = gT^n g^{-1} \cap T^n$ for some $g \in G$ and $(T') \neq (T^n)$.

Proof: By Proposition 2.6.17, the Weyl group $W(T^n)$ is finite and the coefficient of $\Psi(T^n)$ corresponding to (T^n) can be computed as follows (cf. 5.17):

$$\begin{aligned}\chi_c((G/T^n)_{(T^n)}/T^n) &= \chi((G/T^n)^{(T^n)}/T^n) = \chi((G/T^n)^{T^n}/T^n)\\ &= \chi\left((G/T^n)^{T^n}\right) = \chi(G/T^n) = |W(T^n)|.\end{aligned}$$

□

Proposition 5.1.18 tells us what is precisely the coefficient of $\Psi(T^n)$ related to T^n. In general, to compute a coefficient related to an arbitrary (K) in (5.17), one can use the following

Proposition 5.1.19. (Recurrence Formula) *Let T^n be a maximal torus in G, $\psi : T^n \to G$ a natural embedding, and $\Psi : U(G) \to U(T^n)$ the induced homomorphism of the Euler rings. For $(H) \in \Phi(G)$, put*

$$\Psi(H) = \sum_{(K)} n_K(K),$$

where (K)'s stand for the orbit types in the T^n-space G/H, i.e. $K = H' \cap T^n$ with $H' = gHg^{-1}$ for some $g \in G$. Then, for $K = H' \cap T^n$,

$$n_K = \chi\left(\frac{N(K, H')}{H'}/T^n\right) - \sum_{(\tilde{K})>(K)} n_{\tilde{K}}. \tag{5.18}$$

Proof: Put $X := G/H$. Then,

$$X^{(K)}/T^n = \bigcup_{(\tilde{K})\geq(K)} X_{(\tilde{K})}/T^n,$$

which (since T^n is abelian) implies

$$\chi(X^{(K)}/T^n) = \sum_{(\tilde{K})\geq(K)} \chi_c(X_{(\tilde{K})}/T^n) = \sum_{(\tilde{K})\geq(K)} \chi_c(X_{\tilde{K}}/T^n).$$

Therefore,

$$\chi_c(X_K/T^n) = \chi(X^K/T^n) - \sum_{(\tilde{K})>(K)} \chi_c(X_{\tilde{K}}/T^n).$$

To complete the proof, it remains to observe that $X^K/T^n = \frac{N(H'\cap T^n,H')}{H'}/T^n$ (see Proposition 2.4.3) from which (5.18) follows directly. □

Example 5.1.20. Consider the natural embedding $\psi : T^2 := SO(2) \times S^1 \to O(2)\times S^1$, which induces the homomorphism of Euler rings $\Psi : U(O(2)\times S^1) \to U(T^2)$. Using Proposition 5.1.19, one can verify by direct computations that:

$$\begin{aligned}
\Psi(O(2) \times S^1) &= (SO(2) \times S^1), & \Psi(SO(2) \times S^1) &= 2(SO(2) \times S^1)\\
\Psi(D_n \times S^1) &= (\mathbb{Z}_n \times S^1), & \Psi(\mathbb{Z}_m \times S^1) &= 2(\mathbb{Z}_m \times S^1)\\
\Psi(O(2) \times \mathbb{Z}_l) &= (SO(2) \times \mathbb{Z}_l), & \Psi(SO(2) \times \mathbb{Z}_l) &= 2(SO(2) \times \mathbb{Z}_l)\\
\Psi(D_n \times \mathbb{Z}_l) &= (\mathbb{Z}_n \times \mathbb{Z}_l), & \Psi(\mathbb{Z}_m \times \mathbb{Z}_l) &= 2(\mathbb{Z}_m \times \mathbb{Z}_l),\\
\Psi(O(2)^{-,l}) &= (SO(2) \times \mathbb{Z}_l), & \Psi(D_n^{z,l}) &= (\mathbb{Z}_n \times \mathbb{Z}_l)\\
\Psi(SO(2)^{\varphi_m,l}) &= (SO(2)^{\varphi_m,l}) + (SO(2)^{\varphi_{-m},l}), & \Psi(D_{2k}^{d,l}) &= (\mathbb{Z}_{2k}^{d,l})\\
\Psi(\mathbb{Z}_n^{\varphi_m,l}) &= (\mathbb{Z}_n^{\varphi_m,l}) + (\mathbb{Z}_n^{\varphi_{-m},l}), & \Psi(\mathbb{Z}_{2k}^{d,l}) &= 2(\mathbb{Z}_{2k}^{d,l})
\end{aligned}$$

where all the symbols used follow the convention in Appendix A2.1.6.

We conclude this subsection with a brief explanation of how to use the homomorphism $\Psi : U(G) \to U(T^n)$ to compute the multiplication structure in $U(G)$. The knowledge of the Burnside Ring $A_0(G)$ (cf. Subsection 4.1.1), the $A_0(G)$-module $A_1^t(G)$ (cf. Proposition 4.2.5, Remark 5.1.13, Proposition 5.1.3) as well as some ad hoc computations of certain coefficients in the multiplication table for $U(G)$ (cf. Example 5.1.4), may provide one with some information on the structure of $U(G)$. Thus, taking some $(H), (K) \in \Phi(G)$, one can express $(H) * (K)$ as follows

$$(H) * (K) = \sum_{(L)} n_L(L) + \sum_{(L')} x_{L'}(L'), \tag{5.19}$$

where n_L are "known" coefficients while $x_{L'}$ are "unknown". On the other hand, Proposition 5.1.15 allows in principle to completely evaluate the ring $U(T^n)$

(cf. Table A3.19). Since we also know the homomorphism Ψ (cf. Propositions 5.1.18–5.1.19), one has that

$$\Psi((H)) * \Psi((K)) = \sum_{(L'')} n_{L''}(L'') \in U(T^n), \tag{5.20}$$

where all the coefficients $n_{L''}$ are "known". Applying the homomorphism Ψ to (5.19) and comparing the coefficients of the obtained expression with those obtained in (5.20) (related to the same conjugacy classes) leads to a linear system of equations over $\mathbb{Z}$ from which, in principal, it is possible to determine some unknown coefficients in (5.19). However, it might happen that the number of equations in the above linear system is less than the number of unknowns. Summing up, the more partial information on $U(G)$ we have, there is a better chance to compute the remaining coefficients. We will illustrate the described strategy by computing the multiplication table for $U(O(2) \times S^1)$. Take $G := O(2) \times S^1$. Based on the above discussion and the known structure of the Euler ring $U(T^2)$ in Table A3.19, we obtain the Euler ring structure for $U(O(2) \times S^1)$. The multiplication table for $U(O(2) \times S^1)$ is presented in Table A3.20, Appendix A3.

5.2 Equivariant Degree for Gradient G-Maps

Throughout this section, G is a compact Lie group (if not otherwise specified), V is a G-orthogonal representation and $\Omega \subset V$ is an open bounded G-invariant subset.

5.2.1 Construction by K. Gęba and Basic Properties

In this subsection, we follow the construction of the G-equivariant degree for gradient G-maps introduced by K. Gęba in [71] (which is denoted by ∇_G-deg), and discuss some of its basic properties. Based on these properties, we derive an axiomatic definition for ∇_G-deg.

Definition 5.2.1. (i) A map $f : V \to V$ is called a *gradient G-map* if there exists a G-invariant function $\varphi : V \to \mathbb{R}$ of class C^1 such that $f = \nabla\varphi$. Similarly, one can define *gradient G-homotopy*.

(ii) Let $f : V \to V$ be a gradient G-map. The pair (f, Ω) is called a *gradient admissible pair*, if $f(x) \neq 0$ for all $x \in \partial\Omega$. Two gradient admissible pairs (f_0, Ω) and (f_1, Ω) are called *gradient G-homotopic*, if there exists a gradient

G-homotopy $h : [0,1] \times V \to V$ such that $h(0,\cdot) = f_0$, $h(1,\cdot) = f_1$ with $(h(t,\cdot),\Omega)$ being gradient admissible for all $t \in (0,1)$.

Take $x \in V$, put $H := G_x$, and consider the orthogonal decomposition of V

$$V = \tau_x G(x) \oplus W_x \oplus \nu_x, \tag{5.21}$$

where τM denotes the tangent bundle of M, $W_x := \tau_x V_{(H)} \ominus \tau_x G(x)$ and $\nu_x := (\tau_x V_{(H)})^{\perp}$. Suppose $f : V \to V$ is a gradient G-map being differentiable at x and $f(x) = 0$. The derivative $Df(x)$ has a block-matrix form with respect to (5.21)

$$Df(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Kf(x) & 0 \\ 0 & 0 & Lf(x) \end{bmatrix}, \tag{5.22}$$

where $Kf(x) := Df(x)|_{W_x}$ and $Lf(x) := Df(x)|_{\nu_x}$.

Definition 5.2.2. (i) An orbit $G(x)$ is called a *regular zero orbit* of f, if $f(x) = 0$ and $Kf(x) : W_x \to W_x$ (provided by (5.22)) is an isomorphism. Let $E_-(x) \subset W_x$ denote the generalized eigenspace of $Kf(x)$ corresponding to the negative spectrum of $Kf(x)$. Then $\kappa_x := \dim E_-(x)$ is called the *Morse index* of the regular zero orbit $G(x)$. Put

$$i(G(x)) := (-1)^{\kappa(x)}, \tag{5.23}$$

or equivalently,

$$i(G(x)) := \text{sign}\det Kf(x) = \text{sign}\det Df(x)|_{W_x}.$$

(ii) For an open G-invariant subset U of $V_{(H)}$ such that $\overline{U} \subset V_{(H)}$, and a small* $\varepsilon > 0$, put

$$\mathcal{N}(U,\varepsilon) := \{y \in V \ : \ y = x + v, x \in U, v \perp \tau_x V_{(H)}, \|v\| < \varepsilon\},$$

and call it a *tubular neighborhood of type* (H). A gradient G-map $f : V \to V$, $f := \nabla\varphi$ is called (H)-*normal*, if there exists a tubular neighborhood $\mathcal{N}(U,\varepsilon)$ of type (H) such that $f^{-1}(0) \cap \Omega_{(H)} \subset \mathcal{N}(U,\varepsilon)$ and for $y \in \mathcal{N}(U,\varepsilon)$, $y = x + v$, $x \in U, v \perp \tau_x V_{(H)}$,

$$\varphi(y) = \varphi(x) + \frac{1}{2}\|v\|^2,$$

or equivalently,

$$f(y) = f(x) + v.$$

---

* ε is assumed to be sufficiently small that the representation of $y = x + v$ in $\mathcal{N}(U,\varepsilon)$ is unique.

The concept of a generic pair plays an essential role in the construction of the equivariant degree for G-maps presented in [71].

Definition 5.2.3. A gradient admissible pair (f, Ω) is *generic* if there exists an open G-invariant subset $\Omega_o \subset \Omega$ such that

(i) $f|_{\Omega_o}$ is of class C^1;
(ii) $f^{-1}(0) \cap \Omega \subset \Omega_o$;
(iii) $f^{-1}(0) \cap \Omega_o$ is composed of regular orbits of zeros;
(iv) For each (H) with $f^{-1}(0) \cap \Omega_{(H)} \neq \emptyset$, there exists a tubular neighborhood $\mathcal{N}(U, \varepsilon)$ such that f is (H)-normal on $\mathcal{N}(U, \varepsilon)$.

Theorem 5.2.4. (Generic Approximation Theorem, cf. [71]) *For any gradient admissible pair (f, Ω) there exists a generic pair (f_o, Ω) such that (f, Ω) and (f_o, Ω) are gradient G-homotopic.*

Define the equivariant degree for a gradient admissible pair (f, Ω) by

$$\nabla_G\text{-deg}\,(f, \Omega) := \nabla_G\text{-deg}\,(f_o, \Omega) = \sum_{(H) \in \Phi(G)} \mathfrak{n}_H \cdot (H), \tag{5.24}$$

where (f_o, Ω) is a generic approximation pair of (f, Ω) provided by Theorem 5.2.4 and

$$\mathfrak{n}_H := \sum_{(G_{x_i}) = (H)} i(G(x_i)), \tag{5.25}$$

with $G(x_i)$'s being the disjoint orbits of type (H) in $f_o^{-1}(0) \cap \Omega$.

We refer to [71] for the verification that $\nabla_G\text{-deg}\,(f, \Omega)$ is well-defined and satisfies the standard properties expected from a degree.

Now, we are in a position to formulate an alternative axiomatic definition of the degree for gradient G-maps.

Theorem 5.2.5. *Let G be a compact Lie group, $\Omega \subset V$ be an open bounded G-invariant subset and $f : V \to V$ be a gradient G-map. There exists a unique function ∇_G-deg associating to each gradient admissible pair (f, Ω) an element $\nabla_G\text{-deg}\,(f, \Omega) \in U(G)$ such that the following properties are satisfied:*

(P1) (Existence) *If $\nabla_G\text{-deg}\,(f, \Omega) = \sum_{(H)} \mathfrak{n}_H (H)$, is such that $\mathfrak{n}_{H_o} \neq 0$ for some $(H_o) \in \Phi(G)$, then there exists $x_o \in \Omega$ with $f(x_o) = 0$ and $H_o \subset G_x$.*

(P2) (Additivity) *Suppose that Ω_1 and Ω_2 are two disjoint open G-invariant subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then*

$$\nabla_G\text{-deg}\,(f, \Omega) = \nabla_G\text{-deg}\,(f, \Omega_1) + \nabla_G\text{-deg}\,(f, \Omega_2).$$

(P3) (HOMOTOPY) *If $h : [0,1] \times V \to V$ is a gradient G-homotopy being Ω admissible, then*

$$\nabla_G\text{-deg}\,(h_t, \Omega) = \text{constant},$$

where $h_t(\cdot) := h(t,\cdot)$ for $t \in [0,1]$.

(P4) (MULTIPLICATIVITY) *Let V and W be two orthogonal G-representations, (f,Ω) and $(\widetilde{f},\widetilde{\Omega})$ two gradient admissible pairs, where $\Omega \subset V$ and $\widetilde{\Omega} \subset W$. Then*

$$\nabla_G\text{-deg}\,(f \times \widetilde{f}, \Omega \times \widetilde{\Omega}) = \nabla_G\text{-deg}\,(f,\Omega) * \nabla_G\text{-deg}\,(\widetilde{f},\widetilde{\Omega}),$$

where the multiplication '$*$' is taken in the Euler ring $U(G)$.

(P5) (NORMALIZATION) *Suppose (f,Ω) is a generic pair such that $f^{-1}(0) \cap \Omega = G(x_o)$, for some $x_o \in \Omega$ with $H_o := G_{x_o}$. Let $\mathcal{N}(U,\varepsilon)$ be a tubular neighborhood provided by Definition 5.2.3(iv) and $i(G(x_o))$ be defined by (5.23). Then*

$$\nabla_G\text{-deg}\,(f, \mathcal{N}(U,\varepsilon)) = i(G(x_o))(H_o).$$

(P6) (SUSPENSION) *Suppose that W is another orthogonal G-representation and let $\mathcal{O}$ be an open bounded G-invariant neighborhood of 0 in W. Then*

$$\nabla_G\text{-deg}\,(f \times \mathrm{Id}\,, \Omega \times \mathcal{O}) = \nabla_G\text{-deg}\,(f,\Omega).$$

Proof: *Existence.* The existence of ∇_G-deg satisfying (P1)-(P5) is guaranteed by its construction as shown in [71]. The suspension property (P6) is a direct consequence of (P4) and (P5). Indeed, by (P4), we have

$$\nabla_G\text{-deg}\,(f \times \mathrm{Id}\,, \Omega \times \mathcal{O}) = \nabla_G\text{-deg}\,(f,\Omega) * \nabla_G\text{-deg}\,(\mathrm{Id}\,, \mathcal{O}).$$

Since $(\mathrm{Id}\,, \mathcal{O})$ is generic, by (P5),

$$\nabla_G\text{-deg}\,(\mathrm{Id}\,, \mathcal{O}) = i(\{0\})\,(G) = (G),$$

which is the unit element in the ring $U(G)$, thus (P6) follows.
Uniqueness. The uniqueness of ∇_G-deg (f,Ω) is provided by (P5), which leads to its analytic definition (cf. (5.24)—(5.25)). □

We complete this subsection with the following

Lemma 5.2.6. *Let G be a compact Lie group, V an orthogonal G-representation, $\Omega \subset V$ an open bounded G-invariant set and $f : V \to V$ a G-gradient*

Ω-admissible map. Then, for every orbit type (L) in Ω, the map $f^L := f|_{V^L} : V^L \to V^L$ is an Ω^L-admissible $W(L)$-equivariant gradient map. Moreover, if

$$\nabla_G\text{-deg}\,(f, \Omega) = \sum_{(K)\in\Phi(G)} n_K(K),$$

and

$$\nabla_{W(L)}\text{-deg}\,(f^L, \Omega^L) = \sum_{(H)\in\Phi(W(L))} m_H(H),$$

then,

$$n_L = m_{\mathbb{Z}_1}, \tag{5.26}$$

where $\mathbb{Z}_1 = \{e\}$ and $e \in W(L)$ is the identity element.

Proof: By homotopy property of the G-gradient degree, without loss of generality, one can assume that f is generic G-map on Ω. Thus, f^L is generic $W(L)$-map on Ω^L. From the construction of G-gradient degree, formula (5.26) follows. □

5.2.2 Computational Formulae for the G-Gradient Degree for Linear Isomorphisms

The G-gradient degree as described in Subsection 5.2.1, contains a complete topological information on the symmetric properties of zeros of f (cf. [41]). However, the computation of $\nabla_G\text{-deg}\,(f, \Omega)$ is a complicated task, in general. In several important cases from the application viewpoint, it is possible to use the standard linearization techniques so that one can reduce the computation of gradient degrees $\nabla_G\text{-deg}\,(f, \Omega)$ for general G-maps $f : V \to V$ to the computation of $\nabla_G\text{-deg}\,(A, \mathcal{B}_1(V))$ for symmetric linear isomorphisms $A : V \to V$.

By applying suspension property, $\nabla_G\text{-deg}\,(A, B_1(V))$ can be evaluated by $\nabla_G\text{-deg}\,(-\text{Id}\,, B_1(V^-))$, where $V^- \subset V$ is the maximal subspace on which A is negative definite. Since $-\text{Id}$ can be viewed as a product map with respect to the isotypical decomposition of V^-, a further reduction is possible. In terms of the spectra of A, we can write $V^- = \bigoplus_{\mu\in\sigma_-(A)} E(\mu)$, where $\sigma_- := \{\mu \in \sigma \ : \ \mu < 0\}$ is the negative spectrum of A, and $E(\cdot)$ denotes the eigenspace.

Let $\{\mathcal{W}_k\}$, $k = 0, 1, ...$, be the complete list of all irreducible G-representations. Since each $E(\mu)$ is G-invariant, one can consider its G-isotypical decomposition

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_{k_o}(\mu),$$

where $E_k(\mu)$ is modeled on $\mathcal{W}_k$ for $k = 0, 1, 2, \dots, k_o$. Put

$$m_k(\mu) = \dim E_k(\mu)/\dim \mathcal{W}_k, \quad k = 0, 1, 2, \dots, k_o, \tag{5.27}$$

which is called the $\mathcal{W}_k$-*multiplicity* of μ.

By applying the multiplicativity properties, one obtains

$$\nabla_G\text{-deg}\,(A, B_1(V)) = \prod_{\mu \in \sigma_-(A)} \prod_{k=0}^{k_o} (\nabla_G\text{-deg}\,(-\mathrm{Id}\,, B_1(\mathcal{W}_k)))^{m_k(\mu)}, \tag{5.28}$$

where $m_k(\mu)$ is defined by (5.27).

Notice that the values of ∇_G-deg $(-\mathrm{Id}\,, B_1(\mathcal{W}_k))$ contribute as basic building blocks to the value of ∇_G-deg $(A, B_1(V))$, and depend only on the irreducible representation $\mathcal{W}_k$. Therefore, we introduce the following notion:

Definition 5.2.7. We call

$$\mathrm{Deg}_{\mathcal{W}_k} := \nabla_G\text{-deg}\,(-\mathrm{Id}\,, B_1(\mathcal{W}_k)), \tag{5.29}$$

the *basic gradient degree* associated to $\mathcal{W}_k$.

Remark 5.2.8. Observe that the computation of $\mathrm{Deg}_{\mathcal{W}_k}$ can be complicated for an arbitrary G. In the rest of this section, we develop a method for the computation of $\mathrm{Deg}_{\mathcal{W}_i}$, in the case $G = \Gamma \times S^1$, where Γ is a compact Lie group. The main ingredients of the method are

(i) for each $(L) \in \Phi(G)$, the n_L-coefficient of $\mathrm{Deg}_{\mathcal{W}_i}$ can be computed via the $W(L)$-gradient degree of the restriction to V^L (cf. Lemma 5.2.6);

(ii) if $(L) \in \Phi_1^t(G)$, then the computation of the related $W(L)$-gradient degree can be done using a canonical passage via the so-called orthogonal degree (cf. Subsection 5.2.3);

(iii) the computation of basic gradient degree related to the maximal torus-action usually is simple, therefore the remaining (non-twisted) coefficients n_L can be computed using the homomorphism $\Psi : U(G) \to U(T^n)$ and the information obtained for the twisted orbit types.

5.2.3 Passage through Orthogonal Degree for One-dimensional Bi-Orientable Compact Lie Groups

In this subsection, assume that G stands for a one-dimensional bi-orientable compact Lie group. It turns out that, in this case, one can associate to a given G-gradient Ω-admissible map $f : V \to V$, (or more generally, to an orthogonal map (cf. Definition 5.2.9)), a G-equivariant map $\tilde{f} : \mathbb{R} \oplus V \to V$ in such a way that the primary degree of $\tilde{f}$ is intimately connected to $\nabla_G\text{-deg}\,(f, \Omega)$. Observe that in the case $G = \Gamma \times S^1$ with Γ finite, a similar construction was suggested in [152].

We start with the following definition.

Definition 5.2.9. A G-equivariant map $f : V \to V$ is called *G-orthogonal* on Ω, if f is continuous and for all $v \in \Omega$, the vector $f(v)$ is orthogonal to the orbit $G(v)$ at v. Similarly, one can define the notion of a *G-orthogonal homotopy* on Ω.

Clearly, any G-gradient map is orthogonal, however, one can easily construct an orthogonal map which is not G-gradient (cf. [15] for instance).

To associate with an orthogonal map, a G-equivariant map and the corresponding primary degrees, some preliminaries of related G-orbits are necessary.

Take the connected component of $e \in G$, which is a maximal torus T^1 of G. Choose an orientation on T^1 and identify T^1 with S^1. The chosen orientation on S^1 can be extended invariantly on the whole group G. We assume the orientation to be fixed throughout this subsection.

Next, take a vector $v \in V$ and define the diffeomorphism

$$\mu_v : G/G_v \to G(v), \quad \varphi_v(gG_v) := gv. \tag{5.30}$$

Take the decomposition

$$V = V^{S^1} \oplus V', \quad V' := (V^{S^1})^{\perp}. \tag{5.31}$$

If $v \in V^{S^1}$, then $\dim G_x = 1$ so that the orbit $G(x) \cong G/G_x$ is finite and, therefore, admits a natural orientation.

If $v \notin V^{S^1}$, then G_v is a finite subgroup of G, and by bi-orientability of G, both (left and right) actions of G_v preserve the fixed orientation of G.

Therefore, G/G_v has a natural orientation, induced from G. Consequently, the orientation obtained by (5.30) does not depend on a choice of the point v from the orbit $G(v)$ (cf. Remark 2.2.15). More precisely, consider $v \in V$ and the map $\varphi_v : G \to G(v)$ given by

$$\varphi_v(g) = gv, \quad g \in G. \tag{5.32}$$

Clearly φ_v is smooth and $D\varphi_v(1) : \tau_1(G) = \tau_1(S^1) \to \tau_v(G(v))$. Since the total space of the tangent bundle to S^1 can be written as

$$\tau(S^1) = \{(z, \gamma) \in \mathbb{C} \times S^1 : z = it\gamma,\ t \in \mathbb{R}\},$$

a tangent vector to the orbit $G(v)$ can be represented by

$$\tau(v) := D\varphi_v(1)(i) = \lim_{t \to 0} \frac{1}{t}\left[e^{it}v - v\right]. \tag{5.33}$$

Notice that for any $v \in V^{S^1}$, we have $\tau(v) = 0$. Thus, by using the decomposition

$$V = V^{S^1} \oplus V', \quad V' := (V^{S^1})^{\perp}, \tag{5.34}$$

we have that a G-equivariant map $f : V \to V$ is G-orthogonal, if and only if

$$\langle f(x, u), (0, \tau(u)) \rangle = 0,$$

for every $v = (x, u) \in V = V^{S^1} \oplus V'$.

Summing up, in both cases ($v \in V^{S^1}$ and $v \notin V^{S^1}$), $G(v)$ admits a natural orientation, although exhibits different algebraic and topological properties. Hence, given an orthogonal map f, the orbits of $f^{-1}(0)$ belonging to V^{S^1} and those belonging to $V \setminus V^{S^1}$ contribute in equivariant homotopy properties of f in different ways, and one needs to treat these contributions separately.

Definition 5.2.10. Let $f : V \to V$ be a G-orthogonal on Ω. Then, f is called S^1*-normal* on Ω if

$$\exists_{\delta > 0}\ \forall_{x \in \Omega^{S^1}}\ \forall_{u \perp V^{S^1}} \quad \|u\| < \delta \implies f(x + u) = f(x) + u. \tag{5.35}$$

Similarly, one can define the G-orthogonal S^1-normal homotopy on Ω.

We have an S^1-normal approximation theorem.

Theorem 5.2.11. *Suppose that $f : V \to V$ is a G-orthogonal map on Ω. Then, for every $\varepsilon > 0$, there exists a G-orthogonal S^1-normal on Ω map $f_o : V \to V$ such that*

$$\forall_{v\in\overline{\Omega}} \quad \|f(v) - f_o(v)\| < \varepsilon. \tag{5.36}$$

In addition, if f is Ω-admissible, then for $\varepsilon < \min_{v\in\partial\Omega}\|f(v)\|$, f_o is also Ω-admissible. Moreover, f_o is G-orthogonally homotopic to f on Ω via a linear G-orthogonal Ω-admissible homotopy.

Similarly, if $h : [0,1] \times V \to V$ is a G-orthogonal homotopy on Ω, then for every $\varepsilon > 0$, there exists a homotopy $h_o : [0,1] \times V \to V$ which is G-orthogonal on Ω and S^1-normal on Ω such that

$$\forall_{(t,v)\in[0,1]\times\overline{\Omega}} \quad \|h(t,v) - h_o(t,v)\| < \varepsilon. \tag{5.37}$$

In addition, if $h(0,\cdot) =: f_0$ and $h(1,\cdot) =: f_1$ are S^1-normal on Ω, then the homotopy h_o can be constructed in such a way that $h_o(0,\cdot) = f_0$ and $h_o(1,\cdot) = f_1$.

Proof: Consider the decomposition (5.34) of V. For $v \in V$, we write $v = (x,u)$, where $x \in V^{S^1}$ and $u \in V'$. Given $\delta > 0$, define the function $\eta_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\eta_\delta(\rho) := \begin{cases} 0 & \text{if } \rho \le \delta, \\ \frac{\rho-\delta}{\delta} & \text{if } \delta < \rho < 2\delta, \\ 1 & \text{if } \rho \ge 2\delta, \end{cases}$$

(see Figure 5.2.3).

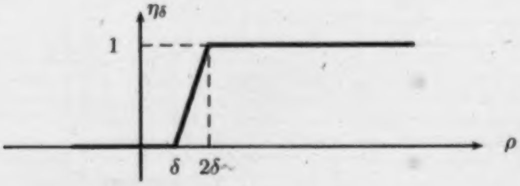

Fig. 5.2. Bump function η_δ

Next, define the map $f_o : V \to V$ by

$$f_o(v) = f_o(x,u) := f(x, \eta_\delta(\|u\|)u) + (1 - \eta_\delta(\|u\|))u. \tag{5.38}$$

By construction, f_o is G-orthogonal and S^1-normal on Ω (with δ as the S^1-normality constant).

Put $\varepsilon_o := \inf\limits_{v\in\partial\Omega}\{\|f(v)\|\}$. By the Ω-admissibility of f, $\varepsilon_o > 0$. We can assume $\varepsilon \le \frac{\varepsilon_o}{2}$. Otherwise, replace ε with $\min\{\varepsilon, \frac{\varepsilon_o}{2}\}$. We claim that for every such $0 < \varepsilon < \frac{\varepsilon_o}{2}$, there exists a proper $\delta > 0$, such that the map f_o defined by (5.38) satisfies

$$\forall_{v\in\overline{\Omega}}\ \ \|f(v) - f_o(v)\| < \varepsilon. \tag{5.39}$$

Since for any $v = (x,u) \in V$ with $\|u\| \ge 2\delta$, $f_o(v) = f(x,u) = f(v)$, it is sufficient to show (5.39) for $v = (x,u) \in \overline{\Omega}$ with $\|u\| < 2\delta$.

By the uniform continuity of f on $\overline{\Omega}$, there exists $\delta_1 > 0$ such that

$$\forall_{v,v'\in\overline{\Omega}}\ \ \|v - v'\| < \delta_1 \ \ \Rightarrow\ \ \|f(v) - f(v')\| < \frac{\varepsilon}{2}.$$

Choose $\delta := \min\{\frac{\delta_1}{2}, \frac{\varepsilon}{2}\} > 0$, thus for all $v = (x,u) \in \overline{\Omega}$ with $\|u\| < 2\delta (< \delta_1)$,

$$\begin{aligned}\|f(v) - f_o(v)\| &= \|f(x,u) - f_o(x,u)\|\\ &= \|f(x,u) - f(x,\eta_\delta(\|u\|)u) - (1-\eta_\delta(\|u\|))u\|\\ &\le \|f(x,u) - f(x,\eta_\delta(\|u\|)u)\| + (1-\eta_\delta(\|u\|))\|u\|\\ &< \frac{\varepsilon}{2} + \delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

By the assumption $\varepsilon \le \frac{\varepsilon_o}{2}$,

$$\forall_{v\in\overline{\Omega}}\ \ \|f(v) - f_o(v)\| < \varepsilon \le \frac{\varepsilon_o}{2}.$$

Thus, for all $v \in \partial\Omega$,

$$\begin{aligned}\|f_o(v)\| &\ge \|f(v)\| - \|f(v) - f_o(v)\|\\ &\ge \varepsilon_o - \frac{\varepsilon_o}{2} = \frac{\varepsilon_o}{2} > 0.\end{aligned}$$

Consequently, f_o is Ω-admissible.

Define the homotopy $h : [0,1] \times V \to V$ by

$$h(t,v) := f(x, tu + (1-t)\eta_\delta(\|u\|)u) + (1-t)(1-\eta_\delta(\|u\|))u,$$

where $t \in [0,1]$. It is clear that $h(0,\cdot) = f_o$ and $h(1,\cdot) = f$. Notice that for $v \in V$ with $\|u\| \geq 2\delta$, $h(t,v) \equiv f(x,u) = f(v)$. To check the Ω-admissibility of $h(t,\cdot)$, it is enough to show that for all $(t,v) \in [0,1] \times \partial\Omega$ with $\|u\| < 2\delta$, we have $\|h(t,v)\| > 0$. Indeed,

$$\begin{aligned}\|h(t,v) - f(v)\| &\leq \|f(x, tu + (1-t)\eta_\delta(\|u\|)u) - f(x,u)\| \\ &\quad + \|(1-t)(1-\eta_\delta(\|u\|))u\| \\ &< \frac{\varepsilon}{2} + \|u\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \leq \frac{\varepsilon_o}{2},\end{aligned}$$

thus

$$\|h(t,v)\| \geq \|f(v)\| - \|h(t,v) - f(v)\| > \varepsilon_o - \frac{\varepsilon_o}{2} = \frac{\varepsilon_o}{2} > 0.$$

Consequently, h is an Ω-admissible homotopy. In order to verify that h is G-orthogonal on $\overline{\Omega}$, we notice that for $(t,v) = (t,x,u) \in [0,1] \times \overline{\Omega}$,

$$\begin{aligned}\langle h(t,x,u), (0,\tau(u))\rangle &= \langle f(x, (t + (1-t)\eta_\delta(\|u\|))u), (0,\tau(u))\rangle \\ &\quad + (1-t)(1-\eta_\delta(\|u\|))\langle u, (0,\tau(u))\rangle\rangle = 0.\end{aligned}$$

The proof for G-orthogonal homotopies is similar. □

We are now in a position to define an orthogonal degree and take a G-orthogonal Ω-admissible map $f : V \to V$. By Theorem 5.2.11, there exists a map $f_o : V \to V$ being G-orthogonal S^1-normal on Ω and G-orthogonally homotopic to f. Consider decomposition (5.34). Since f_o is S^1-normal, there exists $\delta > 0$ such that for all $x \in \Omega \cap V^{S^1}$ and $u \in V'$,

$$f_o(x+u) = f_o(x) + u, \quad \text{provided} \quad \|u\| < \delta.$$

Take the set

$$U_\delta := \{(t,v) \in (-1,1) \times \Omega : v = x + u,\ x \in V^{S^1},\ u \in V',\ \|u\| > \delta\}, \tag{5.40}$$

and define $\widetilde{f}_o : \mathbb{R} \oplus V \to V$ by

$$\widetilde{f}_o(t,v) := f_o(v) + t\tau(v), \quad (t,v) \in \mathbb{R} \oplus V, \tag{5.41}$$

where $\tau(v)$ is given by (5.33). It is clear that $\widetilde{f}_o$ is G-equivariant and U_δ-admissible.

Set $\bar{f}_o := f_o|_{V^{S^1}} : V^{S^1} \to V^{S^1}$, which is G-equivariant and Ω^{S^1}-admissible.

Definition 5.2.12. Let G be a one-dimensional bi-orientable compact Lie group. Consider a G-orthogonal Ω-admissible map $f : V \to V$. Define the *orthogonal G-equivariant degree* $G\text{-Deg}^{o}(f,\Omega)$ of the map f to be an element of $A_0(G) \oplus A_1^{+}(G) \subset A_0(G) \oplus A_1(G) =: U(G)$ given by

$$G\text{-Deg}^{o}(f,\Omega) := \Big(\mathrm{Deg}^0_G(f,\Omega), \mathrm{Deg}^1_G(f,\Omega)\Big), \tag{5.42}$$

where

$$\mathrm{Deg}^0_G(f,\Omega) := G\text{-deg}(\bar{f}_o, \Omega^{S^1}) \in A_0(G), \tag{5.43}$$

and

$$\mathrm{Deg}^1_G(f,\Omega) := G\text{-Deg}\,(\widetilde{f}_o, U_\delta) \in A_1^{+}(G), \tag{5.44}$$

where G-deg stands for the primary G-equivariant degree (cf. Chapter 3).

We claim that the definition (5.42)-(5.44) is independent of the choice of a G-orthogonal S^1-normal approximation f_o. Indeed, assume that $f'_o : V \to V$ is another S^1-normal approximation of f such that

$$\forall_{v\in\overline{\Omega}} \quad \|f(v) - f'_o(v)\| < \varepsilon := \frac{1}{4}\inf_{v\in\partial\Omega}\{\|f(v)\|\}. \tag{5.45}$$

Let δ' be the S^1-normality constant of f'_o, and $U_{\delta'}$ be given by (5.40). Define $F'_o : \mathbb{R} \oplus V \to V$ by

$$F'_o(t,v) := f'_o(v) + t\tau(v), \quad (t,v) \in \mathbb{R} \oplus V.$$

Put $\bar{\delta} := \min\{\delta, \delta'\}$, and define $U_{\bar{\delta}}$ by (5.40). By the excision property of the primary degree, we have

$$G\text{-Deg}\,(F_o, U_{\bar{\delta}}) = G\text{-Deg}\,(F_o, U_\delta),$$

and

$$G\text{-Deg}\,(F'_o, U_{\bar{\delta}}) = G\text{-Deg}\,(F'_o, U_{\delta'}).$$

Also, by (5.45), we have that f_o and f'_o are G-orthogonally homotopic on Ω. In particular, $f_o|_{V^{S^1}}$ and $f'_o|_{V^{S^1}}$ are Γ-homotopic on Ω^{S^1}, thus, by the homotopy property of the primary degree,

$$\Gamma\text{-Deg}(\bar{f}_o, \Omega^{S^1}) = \Gamma\text{-Deg}(\bar{f}'_o, \Omega^{S^1}).$$

Moreover, F_o and F'_o are G-orthogonally homotopic on $U_{\bar{\delta}}$, so by the homotopy property of the primary degree, we have

$$G\text{-Deg}\,(F_o, U_{\bar{\delta}}) = G\text{-Deg}\,(F'_o, U_{\bar{\delta}}).$$

Therefore;

$$G\text{-Deg}\,(F_o, U_{\delta}) = G\text{-Deg}\,(F'_o, U_{\delta'}).$$

In this way, we obtain the following

Theorem 5.2.13. *Suppose that V is an orthogonal representation of the one-dimensional bi-orientable compact Lie group G. For each pair (f, Ω), where $\Omega \subset V$ is an open bounded G-invariant set in V and $f : V \to V$ is a G-orthogonal Ω-admissible map, one can associate the orthogonal G-equivariant degree $G\text{-Deg}^{\,o}(f, \Omega) \in A_0(G) \oplus A_1(G)$ by (cf. (5.42)–(5.44)), which satisfies the following properties:*

(P1) (Existence) *If $G\text{-Deg}^{\,o}(f, \Omega) \neq 0$, i.e. either*

$$\text{Deg}^0_G(f, \Omega) = \sum_{(H) \in \Phi_0(G)} n_H(H) \neq 0,$$

or

$$\text{Deg}^1_G(f, \Omega) = \sum_{(H) \in \Phi_1(G)} n_H(H) \neq 0,$$

meaning that $n_{H_o} \neq 0$ for some $(H_o) \in \Phi_0(G)$ or $(H_o) \in \Phi_1(G)$, then there exists $x_o \in \Omega$ such that $f(x_o) = 0$ and $G_{x_o} \supset H_o$.

(P2) (Additivity) *Suppose that Ω_1 and Ω_2 are two disjoint open G-invariant subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$G\text{-Deg}^{\,o}(f, \Omega) = G\text{-Deg}^{\,o}(f, \Omega_1) + G\text{-Deg}^{\,o}(f, \Omega_2).$$

(P3) (Homotopy) *If $h : [0,1] \times V \to V$ is a G-orthogonal Ω-admissible homotopy, then*

$$G\text{-Deg}^{\,o}(h_t, \Omega) = \text{constant}, \qquad \textit{for all } \ t \in [0,1],$$

where $h_t(v) := h(t, v)$ for $t \in [0,1]$ and $v \in V$.

(P4) (Suspension) *Let W be an orthogonal G-representation and $\mathcal{O} \subset W$ an open bounded G-invariant neighborhood of 0 in W. Then,*

$$G\text{-Deg}^{\,o}(f \times \text{Id}\,, \Omega \times \mathcal{O}) = G\text{-Deg}^{\,o}(f, \Omega).$$

Proof: All the properties are direct consequences of the corresponding properties of the primary degree with one free parameter and primary degree without free parameter (cf. Proposition 3.2.4). □

We complete this subsection with the following result connecting the orthogonal and G-gradient degree in the case G is a compact one-dimensional bi-orientable Lie group.

Proposition 5.2.14. *Let $f : V \to V$ be a G-gradient Ω-admissible map. Then,*

$$\nabla_G\text{-deg}\,(f,\Omega) = \Big(\mathrm{Deg}^0_G(f,\Omega), -\mathrm{Deg}^1_G(f,\Omega)\Big),$$

where $\mathrm{Deg}^0_G(f,\Omega) \in A_0(G)$ is defined by (5.43) and $\mathrm{Deg}^1_G(f,\Omega) \in A_1^+(G)$ is defined by (5.44).

Proof: Without lose of generality, we can assume that f is a generic gradient map on Ω (cf. Theorem 5.2.4). Then, the zero set $f^{-1}(0) \cap \Omega$ is composed of finitely many regular orbits. By the additivity property, we can assume $f^{-1}(0) \cap \Omega$ contains a single orbit $G(x_o)$, being of the orbit type (H_o). Let $\mathcal{N}_o$ be a tubular neighborhood around $G(x_o)$. By the excision property, we have that

$$\nabla_G\text{-deg}\,(f,\Omega) = \nabla_G\text{-deg}\,(f,\mathcal{N}_o).$$

If $x_o \in \Omega^{S^1}$, then $H_o \supset S^1$ is of dimension 1. Thus, the orbit $G(x_o) \simeq G/H_o$ is a finite set, which forces $\tau_{x_o}(G(x_o)) = \{0\}$. Hence, we have the decomposition (cf. (5.21))

$$V = \tau_{x_o}V_{(H_o)} \oplus \nu_x = W_x \oplus \nu_x$$

and the corresponding block matrix

$$Df(x_o) = \begin{bmatrix} Kf(x_o) & 0 \\ 0 & \mathrm{Id} \end{bmatrix}.$$

Consequently,

$$\nabla_G\text{-deg}\,(f,\mathcal{N}_o) = \text{sign}\det Kf(x_o)\cdot(H_o).$$

On the other hand, since f is a generic map on $\mathcal{N}_o$, it is also regular normal on $\mathcal{N}_o$ with (H_o) being the only orbit type. By the elimination property, $\mathrm{Deg}^1_G(f,\Omega) = 0$. To evaluate $\mathrm{Deg}^0_G(f,\Omega)$, observe that the slice S_{x_o} at x_o is isomorphic to $\tau_{x_o}V_{H_o}$ and positively oriented (cf. Definition 2.2.17). Moreover, $\tau_{x_o}V_{H_o} \simeq W_{x_o}$. Indeed, $V_{(H_o)}$ is a disjoint union of $g_iV_{H_o}g_i^{-1}$ for finitely many $g_i \notin H_o$. Therefore,

$$\mathrm{Deg}^0_G(f,\Omega) = G\text{-Deg}\,(f,\mathcal{N}_o^{S^1}) = n_{H_o}\cdot(H_o),$$

where $n_{H_o} = \operatorname{sign}\deg Df(x_o)|_{S_{x_o}} = \operatorname{sign}\det Kf(x_o)$. Hence,

$$\nabla_G\text{-deg}\,(f,\Omega) = (\mathrm{Deg}^0_G(f,\Omega),0).$$

If $x_o \notin \Omega^{S^1}$, then $H_o \not\supset S^1$ is of dimension 0. Thus, the orbit $G(x_o) \simeq G/H_o$ is of dimension 1. Since f is a generic map on the tubular neighborhood $\mathcal{N}_o$, we have

$$\nabla_G\text{-deg}\,(f,\mathcal{N}_o) = \operatorname{sign}\det Kf(x_o)\cdot(H_o).$$

Also, f is S^1-normal on $\mathcal{N}_o$. By the construction, the associated map $F:\mathbb{R}\oplus V \to V$ is regular normal on U_δ (cf. (5.40)–(5.41)). In particular, F is regular normal on $(-\eta,\eta)\times\mathcal{N}_o$ for a small $\eta>0$, which is a tubular neighborhood around $G(0,x_o)$. By the elimination property, $\mathrm{Deg}^0_G(f,\Omega)=0$. By the normalization property, we have

$$\mathrm{Deg}^1_G(f,\Omega) = G\text{-Deg}\,(F,(-\eta,\eta)\times\mathcal{N}_o) = n_{H_o}(H_o).$$

To determine n_{H_o}, observe that

$$\begin{aligned}\mathbb{R}\oplus\tau_{x_o}V_{H_o}' &= \mathbb{R}\oplus\left(\tau_{x_o}V_{H_o}\cap\tau_{x_o}V_{(H_o)}\right)\\ &= \mathbb{R}\oplus\left(\tau_{x_o}V_{H_o}\cap W_{x_o}\right)\oplus\tau_{x_o}(W(H_o)(x_o)),\end{aligned}$$

and the corresponding block matrix is

$$DF(x_o) = \begin{bmatrix} 0 & Kf(x_o) & 0\\ 1 & 0 & 0\end{bmatrix}.$$

Notice that the slice S_{x_o} is isomorphic to $\mathbb{R}\oplus\left(\tau_{x_o}V_{H_o}\cap W_{x_o}\right)$ is positively oriented (cf. Definition 2.2.17). Therefore,

$$n_{H_o} = \operatorname{sign}\det(DF(x_o)|_{S_{x_o}}) = -\operatorname{sign}\det(Kf(x_o)).$$

Hence,

$$\nabla_G\text{-deg}\,(f,\Omega) = (0,-\mathrm{Deg}^1_G(f,\Omega)).$$

□

An immediate consequence of Proposition 5.2.14 is a multiplicativity property of the orthogonal degree, inherited from the same property of the gradient degree.

Corollary 5.2.15. *Let V and W be two orthogonal G-representations, (f,Ω) and $(\widetilde{f},\widetilde{\Omega})$ two gradient admissible pairs, where $\Omega\subset V$ and $\widetilde{\Omega}\subset W$. Then, we have*

(P4)(MULTIPLICATIVITY) *The product map $f\times\widetilde{f}: V\oplus W\to V\oplus W$ is $\Omega\times\widetilde{\Omega}$-admissible, and*

$$G\text{-Deg}^o(f\times\widetilde{f},\Omega\times\widetilde{\Omega}) = G\text{-Deg}^o(f,\Omega) * G\text{-Deg}^o(\widetilde{f},\widetilde{\Omega}),$$

where the multiplication '$*$' is taken in the Euler ring $U(G)$.

A similar result as Proposition 5.2.14 was established in [152], for a special case $G=\Gamma\times S^1$ with Γ being finite.

Corollary 5.2.16. *For $G=\Gamma\times S^1$ with Γ being a finite group, the multiplication in $U(G)$, when restricted to $A_1(G)\times A_1(G)$, is trivial, i.e. for any twisted subgroups $(\mathcal{H}^{\varphi_1,l_1}),(\mathcal{K}^{\varphi_2,l_2})\in\Phi_1(G)$, we have*

$$(\mathcal{H}^{\varphi_1,l_1})\circ(\mathcal{K}^{\varphi_2,l_2})=0.$$

Proposition 5.2.17. *Let G be a bi-orientable 1-dimensional compact Lie group. Identify $U(G)$ with $A_0(G)\oplus A_1(G)$. Then, the Euler ring multiplication table can be represented by Table 5.2.*

$*$	$A_0(G)$	$A_1(G)$
$A_0(G)$	$A_0(G)$-multip	$A_0(G)$-module multip
$A_1(G)$	$A_0(G)$-module multip	0

Table 5.2. $U(G)$-Multiplication Table for One-dimensional Bi-orientable G

Proof: We divide the proof into several claims.

Claim 1. If $(H),(K)\in\Phi_1(G)$, then $(H)*(K)=0$.

It is sufficient to notice that the proof of Proposition 5.1.14 is valid for a 1-dimensional compact Lie group G.

Claim 2. If $(H)\in\Phi_0(G)$, $(K)\in\Phi_1(G)$, then $(H)*(K)\in A_1(G)$.

Take $L\subset G$ such that $(G/H\times G/K)^L\neq\emptyset$. Then, by dimension restrictions, $\dim W(L)=1$ (cf. Proposition 2.4.5(i)), i.e. $(L)\in\Phi_1(G)$.

Claim 3. If $a, c \in A_0(G)$, then $a * c \in A_0(G)$.

Take $(a,b), (c,d) \in A_0(G) \oplus A_1(G) \simeq U(G)$. Let (f_1, Ω_1) and (f_2, Ω_2) be gradient admissible pairs such that

$$\nabla_G\text{-deg}\,(f_1, \Omega_1) = a + b,$$
$$\nabla_G\text{-deg}\,(f_2, \Omega_2) = c + d.$$

By the multiplicativity property, we have

$$\begin{aligned}\nabla_G\text{-deg}\,(f_1 \times f_2, \Omega_1 \times \Omega_2) &= (a + b) * (c + d)\\ &= a * c + a * d + b * c + b * d\\ &= a * c + a * d + b * c,\end{aligned} \tag{5.46}$$

where the last equality is based on the Claim 1.

On the other hand, since G is a bi-orientable 1-dimensional compact Lie group, it is possible to associate the orthogonal degree $G\text{-Deg}^o(f_i, \Omega_i)$ to the pair (f_i, Ω_i) for $i = 1, 2$ (cf. (5.42)–(5.44)). By Proposition 5.2.14, we obtain

$$G\text{-Deg}^o(f_1, \Omega_1) = (a, -b),$$
$$G\text{-Deg}^o(f_2, \Omega_2) = (c, -d).$$

By the multiplicativity property and Claim 1, we have

$$G\text{-Deg}^o(f_1 \times f_2, \Omega_1 \times \Omega_2) = a * c - a * d - b * c. \tag{5.47}$$

Comparing (5.46) with (5.47) and combining Proposition 5.2.14, we conclude that $a * c \in A_0(G)$.

□

5.2.4 Computational Formulae of Gradient $\Gamma \times S^1$-Degree

In this subsection, $G = \Gamma \times S^1$, where Γ is a compact Lie group. It is our interest to establish certain computational formulae for the computations of G-gradient degree. As an example, basic gradient degrees for $G = O(2) \times S^1$ are computed.

Take a G-gradient Ω-admissible map $f : V \to V$. For every orbit type $(L) \in \Phi_1^t(G)$ in Ω, the map $f^L : V^L \to V^L$ is a $W(L)$-equivariant map being admissible on Ω^L. Following the passage described in Subsection 5.2.3, one can associate to the admissible pair (f^L, Ω^L), the orthogonal degree $W(L)\text{-Deg}^o(f^L, \Omega^L)$. Combining Lemma 5.2.6 with Proposition 5.2.14, we obtain

Proposition 5.2.18. *Let $f : V \to V$ be a G-gradient Ω-admissible map, $(L) \in \Phi_1^t(G)$ an orbit type in Ω. Assume*

$$\nabla_G\text{-deg}\,(f, \Omega) = \sum_{(K)\in\Phi(G)} n_K(K),$$

and

$$-W(L)\text{-Deg}^o(f^L, \Omega^L) = \sum_{(H)\in\Phi(W(L))} m_H(H).$$

Then,

$$n_L = m_{\mathbb{Z}_1},$$

where $\mathbb{Z}_1 = \{e\}$ and $e \in W(L)$ is the identity element.

To compute the basic gradient degrees (cf. Definition 5.2.7), we apply Proposition 5.2.18 to the case when f is a linear symmetric isomorphism and Ω is the unit ball in V.

Following the convention for the irreducible representations of $G = \Gamma \times S^1$, we distinguish two types of irreducible G-representations in the list $\{\mathcal{W}_k\}$, $k = 0, 1, 2, ...$ (cf. Table 2.1 for conventions used below).

(i) those, where S^1 acts trivially, which can be identified with irreducible Γ-representations and denoted by $\mathcal{V}_i$, $i = 0, 1, 2, ...$);

(ii) those, where S^1 acts non-trivially defined by an l-folded complex multiplication, which is denoted by $\mathcal{V}_{j,l}$.

Theorem 5.2.19. *Let Γ be a compact Lie group, $G = \Gamma \times S^1$, $\mathcal{V}_i$ be the i-th irreducible G-orthogonal representation with the trivial S^1-action and $\mathcal{V}_{j,l}$ be the (j, l)-th irreducible G-orthogonal representation with a nontrivial S^1-action by an l-folded complex multiplication. Then,*

(a) for $\mathcal{V}_i$,

$$\text{Deg}_{\mathcal{V}_i} = \text{deg}_{\mathcal{V}_i} + T_*;$$

(b) for $\mathcal{V}_{j,l}$*,*

$$\operatorname{Deg}_{\mathcal{V}_{j,l}} = (G) - \deg_{\mathcal{V}_{j,l}} + T_*,$$

where $\deg_{\mathcal{V}_i} \in A_0(G)$*,* $\deg_{\mathcal{V}_{j,l}} \in A_1^t(G)$ *and* $T_* \in A^*(G)$*.*

Proof: (a) This formula follows directly from the construction of G-gradient degree. Indeed, assume

$$\operatorname{Deg}_{\mathcal{V}_j} = \sum_{(L)\in\Phi(G)} n_L\,(L)$$

and

$$\deg_{\mathcal{V}_i} = \sum_{(K)\in\Phi_0(G)} m_K\,(K).$$

Since every generic approximation of $-\mathrm{Id}$ is regular normal, one can easily observe that for $(K) \in \Phi_0(G)$, one has $n_K = m_K$.

(b) This statement is a consequence of Proposition 5.2.18. Indeed, let

$$\deg_{\mathcal{V}_{j,l}} = \sum_{(R)\in\Phi_1^t(G)} m_R\,(R) \qquad \text{and}$$

$$\operatorname{Deg}_{\mathcal{V}_{j,l}} := \nabla_G\text{-deg}(-\mathrm{Id}\,, B_{j,l}) = \sum_{(L)\in\Phi(G)} n_L\,(L),$$

and put $V := V_{j,l}$. Since for $(L) \in \Phi_0(G)$, $V_{(L)} = \{0\}$ if $(L) = (G)$ and $V_{(L)} = \emptyset$ otherwise,

$$n_L = \begin{cases} 1 & \text{if} \quad (L) = (G), \\ 0 & \text{for all } (L) \in \Phi_0(G) \text{ such that } (L) \neq (G). \end{cases} \tag{5.48}$$

To compute the n_L-coefficients of $\operatorname{Deg}_{\mathcal{V}_{j,l}}$ for $(L) \in \Phi_1^t(G)$, observe that the map $-\mathrm{Id}$ is not S^1-normal on V. Take the function $\eta_\delta : \mathbb{R} \to \mathbb{R}$ given by

$$\eta_\delta(\rho) := \begin{cases} 0 & \text{if } \rho < \delta, \\ \frac{\rho-\delta}{\delta} & \text{if } \delta \leq \rho \leq 2\delta, \\ 1 & \text{if } \rho > 2\delta, \end{cases} \tag{5.49}$$

where $\delta > 0$ is chosen to be sufficiently small, and correct $-\mathrm{Id}$ to the S^1-normal map $f_o : V \to V$ by

$$f_o(v) := \eta_\delta(\|v\|)(-v) + (1 - \eta_\delta(\|v\|))v = 1 - 2\eta_\delta(\|v\|)v, \quad v \in V.$$

Next, define the map $\widetilde{f}_o : \mathbb{R} \oplus V \to V$ by formula (5.41). Combining a linear change of variables on V with homotopy and excision property of the twisted degree yields

$$\deg_{\mathcal{V}_{j,l}} = G\text{-Deg}^t(\widetilde{f}_o, U_\delta) \tag{5.50}$$

where U_δ is defined by (5.40).

Take $(L) \in \Phi_1^t(G)$ and put $\widetilde{f}_o^L := \widetilde{f}_o|_{V^L}$. Obviously, the primary degree

$$W(L)\text{-Deg}\,(\widetilde{f}_o^L, U_\delta^L) = \sum_{(K)\in\Phi_1^+(W(L))} \hat{m}_K\,(K) \tag{5.51}$$

is correctly defined. Then, Proposition 4.4 from [15] yields

$$m_L = \hat{m}_{\mathbb{Z}_1}, \tag{5.52}$$

where $\mathbb{Z}_1 = \{e\}$ and $e \in W(L)$ is the identity element.

On the other hand, consider the $W(L)$-equivariant map $-\text{Id}\,|_{V^L}$. By identifying S^1 with the connected component of e in $W(L)$, the above construction utilizing (5.49) can be applied to the map $-\text{Id}\,|_{V^L}$, i.e. put

$$f_*^L(v) := \eta_\delta(\|v\|)(-v) + (1 - \eta_\delta(\|v\|))v = 1 - 2\eta_\delta(\|v\|)v, \quad v \in V^L,$$

and define $\widetilde{f}_*^L : \mathbb{R} \oplus V^L \to V^L$ by

$$\widetilde{f}_*^L(t, v) := f_*^L(v) + t\tau(v) \qquad (v \in V^L).$$

Then, $\widetilde{f}_o^L$ and $\widetilde{f}_*^L$ are homotopic by a U_δ^L-admissible homotopy and

$$W(L)\text{-Deg}\,(\widetilde{f}_o^L, U_\delta^L) = W(L)\text{-Deg}\,(\widetilde{f}_*^L, U_\delta^L).$$

Therefore, by Proposition 5.2.18, $\hat{m}_{\mathbb{Z}_1} = -n_L$ and thus

$$m_L = -n_L. \tag{5.53}$$

By combining (5.48) and (5.53), the conclusion follows. □

For the case $G = \Gamma \times S^1$, where Γ is a finite group, a similar result was established in [152].

Corollary 5.2.20. *Let $G = \Gamma \times S^1$ for Γ being a finite group, , $\mathcal{V}_i$ be the i-th irreducible G-orthogonal representation with the trivial S^1-action and $\mathcal{V}_{j,l}$ be the (j, l)-th irreducible G-orthogonal representation with a nontrivial S^1-action by an l-folded complex multiplication. Then,* *(a) for* $\mathcal{V}_i$,

$$\operatorname{Deg}_{\mathcal{V}_i} = \deg_{\mathcal{V}_i};$$

(b) for $\mathcal{V}_{j,l}$,

$$\operatorname{Deg}_{\mathcal{V}_{j,l}} = (G) - \deg_{\mathcal{V}_{j,l}},$$

where $\deg_{\mathcal{V}_i} \in A_0(G)$ *and* $\deg_{\mathcal{V}_{j,l}} \in A_1^t(G)$.

Remark 5.2.21. In view of Theorem 5.2.19, the computations of $\operatorname{Deg}_{\mathcal{V}_o}$ can be completed by using again the ring homomorphism $\Psi : U(G) \to U(T^n)$ and establishing relations between the unknown coefficients and the values of the gradient degrees.

Let us discuss the gradient basic maps for irreducible T^n-representations. Since T^n is an abelian group every nontrivial irreducible representation of T^n is two-dimensional with only two orbit types (T^n) and (H), where H is a subgroup of T^n. Suppose that $\mathcal{V}_o$ is a nontrivial irreducible representation of T^n. By applying the standard arguments, one can easily construct a generic approximation of $-\mathrm{Id} : \mathcal{V}_o \to \mathcal{V}_o$, which immediately gives that

$$T^n\text{-}\deg(-\mathrm{Id}, B_1(\mathcal{V}_o)) = (T^n) - (H).$$

Consequently, in order to compute the equivariant gradient T^n-degree of $-\mathrm{Id} : V \to V$, where V is an arbitrary orthogonal T^n-representation, it is sufficient to use the simple multiplication formula for the Euler ring $U(T^n)$.

By applying the above results, we obtain the basic gradient degrees for $O(2) \times S^1$ (cf. Appendix A2.3.7).

Remark 5.2.22. Let V be an orthogonal G-representation. Notice that the map $-\mathrm{Id} : V \oplus V \to V \oplus V$ is G-homotopic (in the class of non-gradient G-equivariant maps) to $\mathrm{Id} : V \oplus V \to V \oplus V$, thus

$$G\text{-Deg}(-\mathrm{Id}, B(V \oplus V)) = G\text{-Deg}(\mathrm{Id}, B(V \oplus V)) = (G).$$

On the other hand

$$G\text{-Deg}(-\mathrm{Id}, B(V \oplus V)) = [G\text{-Deg}(-\mathrm{Id}, B(V))]^2 = (G),$$

thus $G\text{-Deg}(-\mathrm{Id}, B(V))$ is an invertible element in $A(G)$. We claim that

$$a := \nabla_G\text{-}\deg(-\mathrm{Id}, B(V \oplus V)) \in U(G),$$

is an invertible element in $U(G)$. Indeed, since $\pi_0(a^2) = \mathbf{1} \in A(G)$, $\mathbf{1} := (G)$, we have

$$a^2 = \mathbf{1} + y_1 + \cdots + y_k, \qquad \text{where} \quad y_i = n_i(G/K_i), \ \ (G/K_i) \in A^*(G).$$

Since the elements y_i are nilpotent (by Proposition 5.1.7) and $U(G)$ is abelian and the element $x := -y_1 - \cdots - y_i$ is nilpotent, thus $a^2 = \mathbf{1} - x$ is invertible with the inverse

$$a^{-2} = \mathbf{1} + x + x^2 + \cdots + x^{n-1},$$

for n sufficiently large, so

$$a^{-1} = a(\mathbf{1} + x + x^2 + \cdots + x^{n-1}).$$

Part II

APPLICATIONS TO SYMMETRIC DIFFERENTIAL SYSTEMS AND VARIATIONAL PROBLEMS

6

Hopf Bifurcation in Symmetric Systems of Functional Differential Equations

In this chapter, we study the occurrence of Hopf bifurcations in a symmetric system of delayed functional differential equations, by means of the (twisted) primary equivariant degree methods. The considered type of system appears in many important models in physics, chemistry, biology, engineering, etc. The existence of symmetries often has an enormous impact on a dynamical process, which may result in formations of various patterns exhibiting particular symmetric properties, such as the onset of turbulence in fluid dynamics (cf. [62]), fluctuations in transmission lines (see [118, 13]), periodic reoccurrence in epidemics (cf. [14]), and traveling waves in neural networks (cf. [181]). The prediction and classification of the displaying and changing patterns in those models are usually of a complex nature.

At the present moment, the standard method to study symmetric Hopf bifurcation is based on a finite-dimensional Lyapunov-Schmidt/Central Manifold theorem reduction and further use of the (equivariant) singularity theory and normal forms (see, Golubitsky [76, 77, 79, 81, 121, 122, 123]). Although very effective, this method is not easy to use as it requires a serious topological/analytical background (e.g. there are serious technical difficulties if the multiplicity of a purely imaginary characteristic root is greater than one). During the 1980s, the method of singularities was already largely developed and successfully applied to bifurcation problems with symmetries (cf. [33, 160, 59, 60, 61, 81, 121, 122, 123]). We should also mention the rational-valued homotopy invariants of "degree type" introduced by F. B. Fuller [67], E. N. Dancer [40] and E. N. Dancer and J. F.Toland [42, 44, 43] as important tools to study the Hopf bifurcation phenomenon (see also [30, 121, 136, 122, 123, 175]). It is our belief that the twisted equivariant degree method (cf. [15, 7, 6, 12, 16]) is able (by taking advantage of computer routines) to handle a huge number of possible symmetry types of the bifurcating periodic solutions and is simple enough to be used by applied mathematicians.

Consider an $\mathbb{R}$-parameterized system of functional differential equations, which is symmetric with respect to a finite* group Γ. Under a reasonable nondegeneracy assumption, for an isolated bifurcation center $(\alpha_o, 0) \in \mathbb{R} \oplus W$ (where W is chosen to be an appropriate functional space), and $i\beta_o$ ($\beta_o > 0$) denotes the purely imaginary characteristic root corresponding to $(\alpha_o, 0)$, we apply the equivariant degree method to analyze and classify the occurrence of symmetric Hopf bifurcation. While the implicit function theorem provides us with a necessary condition for the Hopf bifurcation to take place around $(\alpha_o, 0)$, we formulate a sufficient condition in terms of a topological invariant $\omega(\alpha_o, \beta_o) \in A_1(\Gamma \times S^1)$, defined as a (twisted) primary $\Gamma \times S^1$-equivariant degree. Suppose that

$$\omega(\alpha_o, \beta_o) = n_1(H_1) + n_2(H_2) + \cdots + n_{k_o}(H_{k_o}).$$

The value of the element $\omega(\alpha_o, \beta_o)$ contains information of a symmetric classification of bifurcating branches of non-constant periodic solutions. More precisely, a non-zero coefficient n_k implies the existence of a bifurcating branch of periodic solutions with the orbit types at least (H_k). Moreover, if (H_k) is the so-called *dominating* orbit type (i.e. satisfying certain maximality condition (cf. Definition 6.1.7)), then we can not only predict the existence of bifurcating branches of non-constant periodic solutions with the exact orbit type (H_k), but also establish a lower estimate of the number of bifurcating branches.

To evaluate the invariant $\omega(\alpha_o, \beta_o)$, we derive a computational formula (cf. (6.41)), based on the multiplicativity property of the twisted primary degree (cf. Proposition 4.2.6). As it turns out, the values of the twisted basic degrees as well as the basic degrees without parameters, serve as building blocks for the value of $\omega(\alpha_o, \beta_o)$. The original system of equations contributes through the characteristic operator of the linearized system, in terms of the Morse indices and the so-called *isotypical crossing numbers* (cf. Definition 6.1.4).

The equivariant degree method, which we discussed for the local bifurcation problem study, can be also applied to a *global* Hopf bifurcation problem. For the same $\mathbb{R}$-parametrized system of symmetric functional differential equations, we formulate a similar result to predict an unbounded continuation of symmetric branches of non-constant periodic solutions.

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* This assumption makes the considered group $G = \Gamma \times S^1$ to be bi-orientable automatically. However, the general methodology suggested here for the application to the Γ-symmetric Hopf bifurcation problems is valid for Γ being an arbitrary compact Lie group.

This chapter is organized as follows. In Section 6.1, we present a general setting for studying Γ-symmetric Hopf bifurcation problem for a parametrized system of (delayed) functional differential equations. For an isolated center $(\alpha_o, 0)$ corresponding to a characteristic root $i\beta_o$, a local bifurcation invariant $\omega(\alpha_o, \beta_o)$ is constructed as a twisted $\Gamma \times S^1$-equivariant degree of certain associated map in functional spaces. In Section 6.2, we derive a computational formula for $\omega(\alpha_o, \beta_o)$ using the multiplicatitivity property of the twisted primary degree. In Section 6.3, we provide a procedure to use the Maple©, as an example, the invariants are computed for an S_4-symmetric system of $\mathbb{R}$-parametrized functional differential equations. The table of results is presented in Appendix A4.1 (cf. Table A4.2). In Section 6.4, we study a global Hopf bifurcation problem in the same parametrized system of symmetric functional differential equations. Examples for $\Gamma = D_N, A_4$ will be analyzed.

6.1 Hopf Bifurcation in Symmetric Systems of FDEs

Throughout this chapter, we assume Γ to be a finite group.

Let V be a Γ-orthogonal representation. For a constant $\tau \geq 0$, denote by

$$C_{V,\tau} := \{\varphi : [-\tau, 0] \to V \ : \ \varphi \text{ is continuous}\}, \tag{6.1}$$

which is equipped with the usual supremum norm

$$\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|, \qquad \varphi \in C_{V,\tau}. \tag{6.2}$$

The Γ-action on V induces a natural isometric Banach representation of Γ on the space $C_{V,\tau}$ defined by:

$$(\gamma\varphi)(\theta) := \gamma(\varphi(\theta)), \qquad \gamma \in \Gamma, \ \ \theta \in [-\tau, 0]. \tag{6.3}$$

Given a continuous function $x : \mathbb{R} \to V$ and $t \in \mathbb{R}$, define $x_t \in C_{V,\tau}$ by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0]. \tag{6.4}$$

Consider an $\mathbb{R}$-parametrized family of delayed differential equations

$$\dot{x}(t) = f(\alpha, x_t), \quad t \in \mathbb{R}, \tag{6.5}$$

where $x : \mathbb{R} \to V$ is a continuous function and $f : \mathbb{R} \oplus C_{V,\tau} \to V$ satisfies

(A1) f is continuously differentiable.
(A2) f is Γ-equivariant, where Γ acts trivially on $\mathbb{R}$.
(A3) $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.

In addition, to prevent the steady-state bifurcation, we assume

(A4) $\det D_x f(\alpha, 0)|_V \neq 0$ for all $\alpha \in \mathbb{R}$, where $D_x f$ stands for the partial derivative of f restricted to the space of constant functions $x \in V$.

Definition 6.1.1. A point $(\alpha, x_o) \in \mathbb{R} \oplus V$ is said to be a *stationary point* of (6.5), if $f(\alpha, x_o) = 0$. A stationary point (α, x_o) is called *nonsingular* if the restricted partial derivative $D_x f(\alpha, x_o) : V \to V$ is a linear isomorphism.

By (A3), $(\alpha, 0)$ is a stationary point of (6.5), for all $\alpha \in \mathbb{R}$.

Definition 6.1.2. We say that for $\alpha = \alpha_o$, the system (6.5) has a *Hopf bifurcation* occuring at $(\alpha_o, 0)$ corresponding to the "limit period" $\frac{2\pi}{\beta_o}$, if there exists a family of p_s-periodic non-constant solutions $\{(\alpha_s, x_s(t))\}_{s\in\Lambda}$ (for a proper index set Λ) of (6.5) satisfying the conditions:

(1) The set $K := \overline{\bigcup_{s\in\Lambda}\{(\alpha_s, x_s(t)) : t \in \mathbb{R}\}}$ contains a compact connected set C such that $(\alpha_o, 0) \in C$;
(2) $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\forall (\alpha_s, x_s(t)) \in C \quad \sup_t \|x_s(t)\| < \delta \;\Rightarrow\; \|\alpha_o - \alpha_s\| < \varepsilon \;\text{ and }\; \|p_s - \frac{2\pi}{\beta_o}\| < \varepsilon.$$

6.1.1 Characteristic Equation

Let V^c be a complexification of V, i.e. $V^c := \mathbb{C} \otimes_{\mathbb{R}} V$ (cf. Subsection 2.2.2). Then, V^c has a natural structure of a complex Γ-representation defined by $\gamma(z \otimes x) = z \otimes \gamma x$, for $z \in \mathbb{C}$ and $x \in V$. Suppose that V allows the following Γ-isotypical decomposition (cf. Table 2.1 in Subsection 2.2.2 for conventions)

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r. \tag{6.6}$$

where V_i is modeled on the irreducible Γ-representation $\mathcal{V}_i$. Similarly, V^c has a complex isotypical decomposition

$$V^c = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \tag{6.7}$$

where U_j is modeled on the complex irreducible Γ-representation $\mathcal{U}_j$. Notice that the number s of isotypical components in (6.7), may be different from the number r of isotypical components in (6.6), depending on the type of the irreducible representations $\mathcal{V}_i$ (cf. [27]).

Let (α, x_o) be a stationary point of (6.5). The linearization of (6.5) at (α, x_o) leads to the characteristic equation

$$\det_{\mathbb{C}} \triangle_{(\alpha,x_o)}(\lambda) = 0, \tag{6.8}$$

where

$$\triangle_{(\alpha,x_o)}(\lambda) := \lambda \mathrm{Id} - D_x f(\alpha, x_o)(e^{\lambda \cdot} \cdot)$$

is a complex linear operator from V^c to V^c, with $(e^{\lambda \cdot} \cdot)(\theta, x) = e^{\lambda\theta} x$ and $D_x f(\alpha, x_o)(z \otimes x) = z \otimes D_x f(\alpha, x_o)x$ for $z \otimes x \in V^c$ (cf. [180]). For simplicity, we write

$$\triangle_\alpha(\lambda) := \triangle_{(\alpha,0)}(\lambda).$$

Definition 6.1.3. A solution λ_o to (6.8) is called a *characteristic root* of (6.8) at the stationary point (α, x_o). A nonsingular stationary point (α, x_o) is called a *center*, if (6.8) has a purely imaginary root. We will call (α, x_o) an *isolated center* if it is the only center in some neighborhood of (α, x_o) in $\mathbb{R} \oplus V$.

It is clear that (α, x_o) is a nonsingular stationary point if and only if 0 is not a characteristic root of (6.8) at the stationary point (α, x_o). By (A2) and (A3), the operator $\triangle_\alpha(\lambda) : V^c \to V^c$, $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}$, is Γ-equivariant. Consequently, for each isotypical component U_j is invariant with respect to $\triangle_\alpha(\lambda)$. We put

$$\triangle_{\alpha,j}(\lambda) := \triangle_\alpha(\lambda)|_{U_j}. \tag{6.9}$$

6.1.2 Isotypical Crossing Numbers

We assume that

(A5) There is an isolated center $(\alpha_o, 0)$ for system (6.5) such that (6.8) permits a purely imaginary root $\lambda = i\beta_o$ with $\beta_o > 0$.

Let $B := (0, \delta_1) \times (\beta_o - \delta_2, \beta_o + \delta_2) \subset \mathbb{C}$. By (A5), the constants $\delta_1 > 0$, $\delta_2 > 0$ and $\varepsilon > 0$ can be chosen so small that for every $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$, if there is a characteristic root $u + iv \in \partial B$ at $(\alpha, 0)$, then $u + iv = i\beta_o$ and $\alpha = \alpha_o$.

Note that $\triangle_\alpha(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous in $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$ (see [85]). It follows that $\det_{\mathbb{C}} \triangle_{\alpha_o \pm \varepsilon}(\lambda) \neq 0$ for all $\lambda \in \partial B$. Define for $0 \leq j \leq s$,

$$\mathfrak{t}^{\pm}_{j,1}(\alpha_o,\beta_o) := \deg\left(\det_{\mathbb{C}} \Delta_{\alpha_0\pm\varepsilon,j}(\cdot), B\right), \tag{6.10}$$

where deg stands for the usual Brouwer degree. We can now introduce the following important concept (cf. [53, 114, 116, 118], see also [36, 37, 105, 143, 144, 181]).

Definition 6.1.4. The *$\mathcal{U}_j$-isotypical crossing number* of $(\alpha_o, 0)$ corresponding to the characteristic root $i\beta_o$ is defined as

$$\mathfrak{t}_{j,1}(\alpha_o,\beta_o) := \mathfrak{t}^{-}_{j,1}(\alpha_o,\beta_o) - \mathfrak{t}^{+}_{j,1}(\alpha_o,\beta_o), \tag{6.11}$$

where $\mathcal{U}_j$ is the complex Γ-irreducible representation on which is modeled the isotypical component U_j.

Remark 6.1.5. The crossing number $\mathfrak{t}_{j,1}$ has a very simple interpretation. In the case $\det_{\mathbb{C}}(\Delta_{\alpha,j}(i\beta_o)) = 0$, the number $\mathfrak{t}^{-}_{j,1}$ counts in the set B all the U_j-characteristic roots (with $\mathcal{U}_j$-multiplicity) before α crosses the value α_o, and the number $\mathfrak{t}^{+}_{j,1}$ counts the U_j-characteristic roots in B after α crosses α_o. The difference, which is exactly the number $\mathfrak{t}_{j,1}$, represents the net number of the U_j-characteristic roots which 'escaped' (if $\mathfrak{t}_{j,1}$ is positive) or 'entered' (if $\mathfrak{t}_{j,1}$ is negative) the set B when α was crossing α_o.

For any integer $l > 1$, put

$$\mathfrak{t}_{j,l}(\alpha_o,\beta_o) := \mathfrak{t}_{j,1}(\alpha_o, l\beta_o). \tag{6.12}$$

In order to establish the existence of small amplitude periodic solutions bifurcating from the stationary point $(\alpha_o, 0)$, i.e. the occurrence of the Hopf bifurcation at the stationary point $(\alpha_o, 0)$, and to associate with $(\alpha_o, 0)$ a *local bifurcation invariant*, we apply the standard steps for the degree-theoretical approach described in next two subsections.

6.1.3 Normalization of the Period

By making a change of variable $u(t) = x(\frac{p}{2\pi}t)$, for $t \in \mathbb{R}$, the system (6.5) is transformed to

$$\dot{u}(t) = \frac{p}{2\pi} f(\alpha, u_{t,\frac{2\pi}{p}}), \tag{6.13}$$

where $u_{t,\frac{2\pi}{p}} \in C_{V,\tau}$ is defined by

$$u_{t,\frac{2\pi}{p}}(\theta) = u\big(t + \tfrac{2\pi}{p}\theta\big), \quad \theta \in [-\tau, 0]. \tag{6.14}$$

Clearly, $u(t)$ is a 2π-periodic solution of (6.13) if and only if $x(t)$ is a p-periodic solution of (6.5). Put $\beta := \frac{2\pi}{p}$ and write (6.13) as

$$\dot{u}(t) = \frac{1}{\beta} f(\alpha, u_{t,\beta}). \tag{6.15}$$

6.1.4 Setting in Functional Spaces

We identify $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and introduce the operators

$$L : H^1(S^1; V) \to L^2(S^1; V), \quad Lu(t) = \dot{u}(t), \tag{6.16}$$

$$j : H^1(S^1; V) \to C(S^1; V), \quad j(u(t)) = \widetilde{u}(t), \tag{6.17}$$

$$K : H^1(S^1; V) \to L^2(S^1; V), \quad Ku(t) = \frac{1}{2\pi} \int_0^{2\pi} u(s)\, ds, \tag{6.18}$$

where $H^1(S^1; V)$ (resp. $C(S^1; V)$) denotes the first Sobolev space of 2π-periodic V-valued functions (resp. the space of continuous 2π-periodic V-valued functions equipped with the usual supremum norm). Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$. It can be easily verified that $(L + K)^{-1} : L^2(S^1; V) \to H^1(S^1; V)$ exists.

Define $\mathcal{F} : \mathbb{R}^2_+ \times H^1(S^1; V) \to H^1(S^1; V)$ by

$$\mathcal{F}(\alpha, \beta, u) = (L + K)^{-1}\Big[Ku + \tfrac{1}{\beta} N_f(\alpha, \beta, j(u))\Big], \tag{6.19}$$

where $N_f : \mathbb{R}^2_+ \times C(S^1; V) \to L^2(S^1; V)$ is defined by

$$N_f(\alpha, \beta, v)(t) = f(\alpha, v_{t,\beta}). \tag{6.20}$$

Notice that by the compactness of the embedding map j, the map $\mathcal{F}$ is a compact field on any bounded domain.

Put $W := H^1(S^1; V)$. The space W is an isometric Hilbert representation of the group $\Gamma \times S^1$ with the action given by

$$(\gamma, e^{i\theta})x(t) = \gamma(x(t + \theta)), \quad (\gamma, e^{i\theta}) \in \Gamma \times S^1, \ x \in W. \tag{6.21}$$

The map $\mathcal{F}$ is clearly $\Gamma \times S^1$-equivariant.

Notice that, $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times W$ is a 2π-periodic solution of (6.15) if and only if $u = \mathcal{F}(\alpha, \beta, u)$. Consequently, the occurrence of a Hopf bifurcation at $(\alpha_o, 0)$ for the equation (6.5) is equivalent to a bifurcation of 2π-periodic solutions of (6.15) from $(\alpha_o, \beta_o, 0)$ for some $\beta_o > 0$. On the other hand, if a bifurcation at $(\alpha_o, \beta_o, 0) \in \mathbb{R}^2_+ \times W$ takes place in (6.15), then we *necessarily* have that the operator $\mathrm{Id} - D_u\mathcal{F}(\alpha_o, \beta_o, 0) : W \to W$ is not an isomorphism, or equivalently, $il\beta_o$, for some $l \in \mathbb{N}$, is a purely imaginary characteristic root of $(\alpha_o, 0)$, i.e. $\det_{\mathbb{C}} \Delta_{\alpha_o}(il\beta_o) = 0$.

6.1.5 Local $\Gamma \times S^1$-Invariant

It is convenient to identify $\mathbb{R}^2_+$ with a subset of $\mathbb{C}$, i.e. an element $(\alpha, \beta) \in \mathbb{R}^2_+$ will be written as $\lambda = \alpha + i\beta$, and put $\lambda_o = \alpha_o + i\beta_o$. By (A5), $(\alpha_o, 0)$ is an isolated center, which implies that there exists $\delta > 0$ such that $\mathrm{Id} - D_u\mathcal{F}(\lambda, 0) : W \to W$ is an isomorphism for $0 < |\lambda - \lambda_o| \le \delta$. Consequently, by the implicit function theorem, there exists ρ, $0 < \rho < \min\{1, \delta\}$, such that $u - \mathcal{F}(\lambda, u) \neq 0$ for (λ, u) with $|\lambda - \lambda_o| = \delta$ and $0 < \|u\| \le \rho$.

Define the subset $\Omega \subset \mathbb{R}^2_+ \times W$ by

$$\Omega := \Big\{(\lambda, u) \in \mathbb{R}^2_+ \times W : |\lambda - \lambda_o| < \delta,\ \|u\| < \rho\Big\} \tag{6.22}$$

and put

$$\partial_0 := \overline{\Omega} \cap \left(\mathbb{R}^2_+ \times \{0\}\right) \quad \text{and} \quad \partial_\rho := \{(\lambda, u) \in \overline{\Omega} : \|u\| = \rho\}.$$

Following the standard degree theory treatment of the bifurcation phenomenon (see, for instance, [101, 96]), take an *auxiliary* function $\varsigma : \overline{\Omega} \to \mathbb{R}$, which is G-invariant and satisfies the conditions

$$\begin{cases} \varsigma(\lambda, u) > 0 & \text{for } (\lambda, u) \in \partial_\rho, \\ \varsigma(\lambda, u) < 0 & \text{for } (\lambda, u) \in \partial_0. \end{cases}$$

Such a function ς can be easily constructed, for example,

$$\varsigma(\lambda, u) = |\lambda - \lambda_o|(\|u\| - \rho) + \|u\| - \frac{\rho}{2}; \quad (\lambda, u) \in \overline{\Omega}. \tag{6.23}$$

Define the map $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \oplus W$, $\pi(\lambda, u) = u$, by

$$\mathfrak{F}_\varsigma(\lambda, u) = \Big(\varsigma(\lambda, u), u - \mathcal{F}(\lambda, u)\Big), \quad (\lambda, u) \in \overline{\Omega}, \tag{6.24}$$

which is an Ω-admissible $\Gamma \times S^1$-equivariant compact field.

Following the standard lines, one can extend the equivariant degree theory to parametrized equivariant compact fields on Hilbert isometric G-representations (cf. [15] for more details). We use the same symbol to denote the extended equivariant degree.

Definition 6.1.6. Let $\Omega \subset \mathbb{R}^2_+ \times W$ be defined by (6.22) and $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \oplus W$ be defined by (6.24). We call

$$\omega(\lambda_o) := G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) \in A_1(G), \tag{6.25}$$

the *local* $\Gamma \times S^1$*-invariant* for the Γ-symmetric Hopf bifurcation of the system (6.5) at $(\lambda_o, 0)$.

6.1.6 Dominating Orbit Types

The concept of dominating orbit types plays an important role in obtaining a lower estimate of bifurcating branches.

Definition 6.1.7. An orbit type (H) in W is called *dominating*, if (H) is a maximal orbit type in the class of all φ-twisted 1-folded orbit types in W.

Remark 6.1.8. Assume that there is a solution $u_o \in W$ to (6.15) such that $G_{u_o} \supset H_o$. If (H_o) is a dominating orbit type in W with the form $H_o = K^\varphi$ for $K \subset \Gamma$, then $(G_{u_o}) = (K^{\varphi,l})$ for an integer $l \geq 1$. In this case, the G-orbit $G(u_o)$ is composed of exactly $|G/G_{u_o}|_{S^1}$ different periodic functions, where $|G/G_{u_o}|_{S^1}$ denotes the number of S^1-orbits in G/G_{u_o}. In turn, $|G/G_{u_o}|_{S^1}$ can be evaluated by $|\Gamma/K|$, where $|X|$ stands for the number of elements in X. Moreover, let x_o be a p-periodic solution to (6.5) canonically corresponding to u_o with $G_{u_o} = K^{\varphi,l}$. It follows that x_o is also a $\frac{p}{l}$-periodic solution to (6.5). The pair $(x_o, \frac{p}{l})$ canonically determines an element $u_o' \in W$ being a solution to (6.15) (for $\alpha = \alpha_o$ and some β') satisfying the condition $G_{u_o'} = H_o$. In this way, we obtain that (6.5) has at least $|\Gamma/K|$ different periodic solutions with the orbit type *precisely* (H_o).

6.1.7 Sufficient Condition for Symmetric Hopf Bifurcation

Following the same lines as in the proof of Theorem 3.2 from [53] (see also [114] and [12]), one can easily establish

Theorem 6.1.9. *Given system (6.5), assume conditions (A1)–(A5) to be satisfied. Take $\mathcal{F}$ defined by (6.19) and construct Ω according to (6.22). Let $\varsigma : \overline{\Omega} \to \mathbb{R}$ be a G-invariant auxiliary function (see (6.23)) and let $\mathfrak{F}_\varsigma$ be defined by (6.24).*

(i) Assume $\omega_o(\lambda_o) = G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) \neq 0$, i.e.

$$G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) = \sum_{(H)} n_H(H), \qquad \text{and } n_{H_o} \neq 0 \tag{6.26}$$

for some $(H_o) \in \Phi_1(G)$. Then, there exists a branch of non-trivial solutions to (6.5) bifurcating from the point $(\alpha_o, 0)$ (with the limit frequency $l\beta_o$ for some $l \in \mathbb{N}$). More precisely, the closure of the set composed of all non-trivial solutions $(\lambda, u) \in \Omega$ to (6.15), i.e.

$$\overline{\{(\lambda, u) \in \Omega : \mathfrak{F}(\lambda, u) = 0, \ u \neq 0\}}$$

contains a compact connected subset C such that

$$(\lambda_o, 0) \in C \quad \text{and} \quad C \cap \partial_r \neq \emptyset, \quad C \subset \mathbb{R}^2_1 \times W^{H_o},$$

$(\lambda_o = \alpha_o + i\beta_o)$ which, in particular, implies that for every $(\alpha, \beta, u) \in C$ we have $G_u \supset H_o$.

(ii) If, in addition, (H_o) is a dominating orbit type in W, then there exist at least $|G/H_o|_{S^1}$ different branches of periodic solutions to the equation (6.5) bifurcating from $(\alpha_o, 0)$ (with the limit frequency $l\beta_o$ for some $l \in \mathbb{N}$). Moreover, for each (α, β, u) belonging to these branches of (non-trivial) solutions one has $(G_u) = (H_o)$ (considered in the space W) (cf. Remark 6.1.8).

Remark 6.1.10. It is usually the case that there are more than one dominating orbit types in W contributing to the lower estimate of all bifurcating branches of solutions. An additional contribution may come from a nontrivial (K)-term for non-dominating orbit type, such that $n_H = 0$ for all dominating orbit types $(H) > (K)$. Then, we can also predict the existence of multiple branches by analyzing all the dominating orbit types (H) larger than (K). However, in such case, the exact symmetry of the branches can not be determined.

6.2 Computation of the Local $\Gamma \times S^1$-invariant

We use a sequence of reductions based on the properties of the twisted primary degree (cf. Proposition 4.2.7), to establish an effective computational formula for $\omega(\lambda_o)$.

6.2.1 Linearization Procedure

Let $\Omega \subset \mathbb{R}^2_+ \times W$ be given by (6.22). Define another auxiliary function $\widetilde{\varsigma} : \overline{\Omega} \to \mathbb{R}$ by (which is a slight modification of (6.23))

$$\widetilde{\varsigma}(\lambda, u) = |\lambda - \lambda_o|(\|u\| - \rho) + \|u\| + \frac{\delta}{2}\rho, \quad (\lambda, u) \in \overline{\Omega}.$$

By direct verification, $\mathfrak{F}_\varsigma$ and $\mathfrak{F}_{\widetilde{\varsigma}}$ are G-homotopic on Ω by a linear homotopy. Thus, we have

$$G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) = G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega),$$

where $\mathfrak{F}_{\widetilde{\varsigma}} : \overline{\Omega} \to \mathbb{R} \oplus W$ is defined by

$$\mathfrak{F}_{\widetilde{\varsigma}}(\lambda, u) = \Big(\widetilde{\varsigma}(\lambda, u), u - \mathcal{F}(\lambda, u)\Big). \tag{6.27}$$

An advantage of $\widetilde{\varsigma}$ over ς seems to be that it is positive, for λ very close to λ_o in Ω. More precisely, for $|\lambda - \lambda_o| \le \frac{\delta}{4}$ and $\|u\| \le \rho$, we have

$$\widetilde{\varsigma}(\lambda, u) = \|u\| + \frac{\delta}{2}\rho - |\lambda - \lambda_o|(\rho - \|u\|) \ge \frac{\delta}{2}\rho - \frac{\delta}{4}\rho = \frac{\delta}{4}\rho > 0.$$

Put

$$\Omega_1 := \Big\{(\lambda, u) \in \mathbb{R}^2_+ \times W : \|u\| < \rho,\ \frac{\delta}{4} < |\lambda - \lambda_o| < \delta\Big\}. \tag{6.28}$$

By excision property, we obtain

$$G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega) = G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega_1).$$

Define the operator

$$a(\lambda, 0) := \text{Id} - D_u\mathcal{F}(\lambda, 0) : W \to W, \tag{6.29}$$

which is a linearization of the second component of $\mathfrak{F}_\varsigma$ with respect to u at $(\lambda, 0)$ (cf. 6.27), and $A_\varsigma : \overline{\Omega}_1 \to \mathbb{R} \oplus W$ by

$$A_\varsigma(\lambda, u) := (\widetilde{\varsigma}(\lambda, u), a(\lambda, 0)u) \tag{6.30}$$

which is clearly Ω_1-admissible. By homotopy property, we have

$$G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega_1) = G\text{-Deg}\,(A_\varsigma, \Omega_1).$$

6.2.2 Reduction Through Isotypical Decompositions

To take advantage of the multiplicativity properties of both the primary degree without parameters (cf. Proposition 4.1.4) and the twisted primary degree (cf. Proposition 4.2.6), we carry out a series of reduction based on the isotypical decompositions of W.

Viewed as an S^1-orthogonal representation, W admits an S^1-isotypical decomposition (cf. [15])

$$W = W^{S^1} \oplus \overline{\bigoplus_{l=1}^{\infty} W_l}, \tag{6.31}$$

where $W^{S^1} \simeq V$ is the subspace of the constant functions in W and each $W_l \simeq V^c$ is a complex Γ-representation defined by

$$W_l = \{e^{ilt}(x_n + iy_n) : x_n, y_n \in V\}, \quad l = 1, 2, \dots \tag{6.32}$$

Consider the linear operator $a(\lambda, 0) : W \to W$ restricted to each isotypical component in (6.31). By direct verification, we have

$$\begin{aligned} a(\lambda, 0)|_{W^{S^1}} &= -\frac{1}{\beta} D_x f(\alpha, 0), \\ a(\lambda, 0)|_{W_l} &= \frac{1}{il\beta}\, \Delta_\alpha\,(il\beta). \end{aligned} \tag{6.33}$$

Put $W_o := \overline{\bigoplus_{l=1}^{\infty} W_l}$. Define $\Omega_o \subset \mathbb{R}^2 \oplus W_o$ by

$$\begin{aligned} \Omega_o &:= \Omega_1 \cap (\mathbb{R}^2 \oplus W_o) \\ &= \left\{(\lambda, u) \in \mathbb{R}^2_+ \times W_o : \|u\| < \rho,\ \frac{\delta}{4} < |\lambda - \lambda_o| < \delta\right\} \end{aligned} \tag{6.34}$$

and a map $A_o : \overline{\Omega}_o \to \mathbb{R} \oplus W_o$ by

$$A_o(\lambda, u_o) = (\widetilde{\varsigma}(\lambda, u_o), a(\lambda, 0)u_o), \quad (\lambda, u_o) \in \overline{\Omega}_o,$$

which is clearly a G-equivariant Ω_o-admissible compact field. Put

$$\overline{A} := a(\lambda_o, 0)|_{W^{S^1}}.$$

By (A4), $\overline{A}$ is a (symmetric) linear isomorphism on V, thus is $B_1(V)$-admissible.

Notice that the map A_ς is homotopic to the product map $\overline{A} \times A_o$ on $B_1(V) \times \Omega_o$. By multiplicativity property of twisted primary degree (cf. Proposition 4.2.6), we have

$$G\text{-Deg}\,(A_\varsigma, \Omega_1) = \Gamma\text{-Deg}(\overline{A}, B_1(V)) \circ G\text{-Deg}\,(A_o, \Omega_o), \tag{6.35}$$

where $\circ$ is the multiplication taken in $A_0(\Gamma)$-module $A_1(G)$.

Computations of Γ-Deg$(\overline{A}, B_1(V))$

To compute Γ-Deg$(\overline{A}, B_1(V))$, we adopt the computational formula for the primary degree without parameters for linear isomorphisms (cf. Subsection 4.1.3, (4.5)). Thus, we have

$$\Gamma\text{-Deg}\,(\overline{A}, B_1(V)) = \prod_{\mu \in \sigma_-(\overline{A})} \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{m_i(\mu)}, \tag{6.36}$$

where the multiplication is taken in the Burnside ring $A_0(\Gamma)$.

Computations of G-Deg (A_o, Ω_o)

To evaluate G-Deg (A_o, Ω_o), consider further isotypical decomposition of W_o. Since each $W_l \simeq V^c$, the isotypical decomposition (6.7) of V^c induces the corresponding G-isotypical decomposition of W_l

$$W_l = V_{0,l} \oplus V_{1,l} \oplus \cdots \oplus V_{s,l}, \tag{6.37}$$

where each $V_{j,l}$ is modeled on the irreducible representation $\mathcal{V}_{j,l}$ (cf. Table 2.1 for convention). The linear operator $a(\lambda, 0)$ defined by (6.33), when restricted on each $V_{j,l}$ gives

$$a(\lambda,0)|_{V_{j,l}} = \frac{1}{il\beta}\,\triangle_{\alpha,j}\,(il\beta), \tag{6.38}$$

where $\triangle_{\alpha,j}$ is defined by (6.9).

Define $\Omega_{j,l} := \Omega_o \cap V_{j,l}$ and $A_{j,l} : \overline{\Omega}_{j,l} \to \mathbb{R} \oplus V_{j,l}$ by

$$A_{j,l}(\lambda,u) := (\widetilde{\varsigma}(\lambda,u), a(\lambda,0)u), \quad (\lambda,u) \in \overline{\Omega}_{j,l}.$$

By the splitting lemma (cf. Lemma 3.3.4), we obtain

$$\begin{aligned} G\text{-Deg}\,(A_o,\Omega_o) &= \sum_{j,l} G\text{-Deg}\,(A_{j,l},\Omega_{j,l}) \\ &= \sum_{j,l} \deg\left(\det_{\mathbb{C}} \circ\, a(\cdot,0)|_{V_{j,l}}, S^1\right) \cdot \deg_{\mathcal{V}_{j,l}}, \end{aligned} \tag{6.39}$$

where $a(\cdot,0)(\lambda) := a(\lambda,0)$ (cf.(6.38)), 'deg' stands for the Brouwer degree and $\deg_{\mathcal{V}_{j,l}}$ is the twisted basic degree of $\mathcal{V}_{j,l}$ (cf. Definition 4.2.8). Moreover, each coefficient in (6.39) can be evaluated by (cf. [15])

$$\deg\left(\det_{\mathbb{C}} \circ\, a(\lambda,0)|_{V_{j,l}}, S^1\right) = \mathfrak{t}_{j,l}.$$

Therefore, we have

$$G\text{-Deg}\,(A_o,\Omega_o) = \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o,\beta_o)\deg_{\mathcal{V}_{j,l}}, \tag{6.40}$$

where the summation is taken over only finitely many (j,l)'s. Indeed, $\mathfrak{t}_{j,l} = 0$ for all l such that $il\beta_o$ is not a characteristic root of (6.8) at the stationary point $(\alpha_o,0)$.

Combining (6.35)—(6.36) and (6.40), we obtain

$$G\text{-Deg}\,(A_{\widetilde{\varsigma}},\Omega_1) = \prod_{\mu\in\sigma_-(\overline{A})} \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)} \cdot \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o,\beta_o)\deg_{\mathcal{V}_{j,l}}. \tag{6.41}$$

6.3 Computational Example

We consider the following system of delayed differential equations

$$\frac{d}{dt}x(t) = -\alpha x(t) + \alpha H(x(t)) \cdot C(G(x(t-1))), \tag{6.42}$$

where $x := (x^1, x^2, \dots, x^n)^T$, $H(x) := (h(x^1), h(x^2), \dots, h(x^n))^T$, $G(x) := (g(x^1), g(x^2), \dots, g(x^n))^T$, and the product '$\cdot$' is defined on the vectors by component-wise multiplication.

Assume that

(G1) The functions $h, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, $h(t) \neq 0$ for all $t \in \mathbb{R}$, $g(0) = 0$, $g'(0) > 0$ and C is a symmetric $n \times n$-matrix, which commutes with an orthogonal Γ-representation.

6.3.1 Characteristic Values

Consider the linearization of the system (6.42) at $(\alpha, 0)$ by

$$\frac{d}{dt}x(t) = -\alpha x(t) - \alpha h(0)g'(0)C(x(t-1)), \tag{6.43}$$

and put

$$\eta := h(0)g'(0). \tag{6.44}$$

Thus, the assumption (A4) amounts to

$$\prod_{i=1}^{n} \Big[-\alpha - \alpha\eta\xi_i \Big] \neq 0. \tag{6.45}$$

Moreover,

$$\triangle_\alpha(\lambda) = (\lambda + \alpha)\mathrm{Id} + \alpha\eta e^{-\lambda}C,$$

and a number $\lambda \in \mathbb{C}$ is a characteristic root of (6.8) at the stationary point $(\alpha_o, 0)$ if and only if

$$\det{}_{\mathbb{C}} \triangle_\alpha(\lambda) = \prod_{i=1}^{n} \Big[\lambda + \alpha + \alpha\eta\xi_i e^{-\lambda} \Big] = 0, \tag{6.46}$$

where $\xi_1, \xi_2, \dots, \xi_n$ are the eigenvalues of the matrix C.

For $\xi_o \in \sigma(C)$, rewrite $\lambda + \alpha + \alpha\eta\xi_o e^{-\lambda} = 0$ into the system

$$\begin{cases} u + \alpha + \alpha\eta\xi_o e^{-u}\cos v = 0 \\ v - \alpha\eta\xi_o e^{-u}\sin v = 0, \end{cases} \tag{6.47}$$

where $\lambda = u + iv$. Solving for $\lambda = i\beta_o$, we arrive at the following relations between α and β (cf. [15]),

$$\begin{cases} \cos\beta = -\frac{1}{\eta\xi_o} \\ \sin\beta = \frac{1}{\alpha\eta\xi_o}\beta, \end{cases} \tag{6.48}$$

for a nonzero $\xi_o \in \sigma(C)$. If $\left|\frac{1}{\xi_o\eta}\right| < 1$, then there exists $\beta_o \in (0, \pi]$ such that $\cos\beta_o = -\frac{1}{\eta\xi_o}$, and it is also possible to find a unique $\alpha_o = -\beta_o \cot\beta_o$. Therefore, we assume that

(G2) $|\frac{1}{\xi\eta}| < 1$ for all non-zero $\xi \in \sigma(C)$.

6.3.2 Isotypical Crossing Numbers

To determine the value of the crossing number associated with a purely imaginary characteristic root $\lambda_o = i\beta_o$, we carry out an implicit differentiation to compute $\frac{d}{d\alpha}u(\alpha)$, where u is viewed as a function of α (cf. (6.47)). By direct calculation, we obtain

$$\frac{d}{d\alpha}u|_{\alpha=\alpha_o} = \frac{\beta_o^2}{\alpha_o((\alpha_o+1)^2+\beta_o^2)}, \tag{6.49}$$

thus

$$\operatorname{sign} \frac{d}{d\alpha}u|_{\alpha=\alpha_o} = \operatorname{sign} \alpha_o. \tag{6.50}$$

Therefore, we have (cf. [15])

$$\begin{array}{lll} \text{if} \quad \alpha_o > 0 & \text{then} & \mathfrak{t}_{j,1}(\alpha_o, \beta_o) = -m_j(i\beta_o) \\ \text{if} \quad \alpha_o < 0 & \text{then} & \mathfrak{t}_{j,1}(\alpha_o, \beta_o) = m_j(i\beta_o), \end{array}$$

where $m_j(i\beta_o)$ is the multiplicity of 0 viewed as an eigenvalue of the characteristic operator $\triangle_{\alpha_o,j}(i\beta_o)$, i.e.

$$m_j(i\beta_o) = \dim\ker\triangle_{\alpha_o,j}(i\beta_o)/\dim\mathcal{V}_{j,1}. \tag{6.51}$$

To have a definiteness of the sign of α_o, we assume that

(G3) $h(0) > 0$.

Then, we have $\eta = h(0)g'(0) > 0$ and thus from (6.48), it follows that $\operatorname{sign}\alpha_o = \operatorname{sign}\xi_o$. Therefore,

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) = -\operatorname{sign}(\xi_o)m_j(i\beta_o). \tag{6.52}$$

6.3.3 Computational Scheme

The local bifurcation invariant $\omega(\lambda_o)$ defined by (6.25) provides a complete description of the symmetric Hopf bifurcation at $(\alpha_o, 0)$ (cf. Theorem 6.1.9(i)). However, instead of computing the entire value of $\omega(\lambda_o)$ according to (6.41), for simplicity, we will restrict our computations to the coefficients n_{H_o} for the twisted 1-folded orbit types (H_o), and denote the corresponding part of $\omega(\lambda_o)$ by $\omega(\lambda_o)_1$. Clearly, $\omega(\lambda_o)_1$ can be computed by

$$\omega(\lambda_o)_1 = \prod_{\mu\in\sigma_-(\overline{A})} \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)} \cdot \sum_j \mathfrak{t}_{j,1}(\alpha_o, \beta_o) \deg_{\mathcal{V}_{j,1}}. \tag{6.53}$$

Based on the discussion in Subsection 6.3.1–6.3.2, we summarize a computational scheme to conduct efficient computation of $\omega(\lambda_o)_1$.

- ► Take a non-zero $\xi_o \in \sigma(C)$ and find a solution (α_o, β_o) to the system (6.48). In this way, we obtain an isolated center $(\alpha_o, 0)$ and a purely imaginary root $i\beta_o$ such that $\det_{\mathbb{C}} \Delta_{\alpha_o}(i\beta_o) = 0$.
- ► Determine $\ker \Delta_{\alpha_o,j}(i\beta_o)$ by taking $\ker \Delta_{\alpha_o}(i\beta_o) \cap V_{j,l}$ and compute the multiplicity number $m_j(i\beta_o)$ by (6.51).
- ► Evaluate the isotypical crossing numbers by (6.52).
- ► Identify $\sigma_-(\overline{A})$ by $\sigma_-(\overline{A}) = \{\mu \,:\, \alpha - \alpha\eta\xi < 0,\ \xi \in \sigma(C)\}$. For each $\mu \in \sigma_-(\overline{A})$, take the corresponding $\xi \in \sigma(C)$ and compute the $\mathcal{V}_i$-multiplicity of ξ by $m_i(\mu) := \dim\left(E(\xi) \cap V_i\right)/\dim \mathcal{V}_i$.
- ► Insert the numbers $m_i(\mu)$ and $\mathfrak{t}_{j,1}(\alpha_o, \beta_o)$ into the formula (6.53), together with the basic degrees prepared in the catalogue (cf. Appendix A2).

6.3.4 Usage of Maple© Routines

We will briefly describe how to use the Maple© procedure to obtain immediate values of $\omega(\lambda_o)_1$, especially what data need to be prepared in advance for the input and in which format.

In all the computational examples considered in this thesis, the following conditions verify automatically:

(R1) The decomposition (6.6) contains isotypical components modeled only on irreducible representations of *real* type. In particular, $r = s$ in (6.6)–(6.7).

(R2) For each $\xi_o \in \sigma(C)$ there exists a *single* isotypical component V_j in (6.6) which contains the eigenspace $E(\xi_o)$ completely.

To simplify the input data for the computations of Γ-Deg$(\overline{\mathcal{A}}, B_1(V))$, observe that $(\deg_{\mathcal{V}})^2 = (\Gamma)$ for any basic degrees $\deg_{\mathcal{V}}$ without parameters. Therefore, we define the sequence $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r)$ by

$$\varepsilon_i = \sum_{\mu \in \sigma_-(\overline{\mathcal{A}})} m_i(\mu) \pmod 2.$$

Then, the formula (6.36) can be reduced to

$$\Gamma\text{-Deg}(\bar{\mathfrak{F}}, \mathcal{B}) = \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i}.$$

On the other hand, under the condition (R2), we have

$$\omega(\alpha_0, \beta_o)_1 = \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i} \cdot \left(-\operatorname{sign}(\xi_o^j)\right) m_j(i\beta_o) \deg_{\mathcal{V}_{j,1}},$$

where the notation $\xi_o^j = \xi_o$ is to emphasize the index j such that $E(\xi_o) \subset V_j$ (cf. (R2)). For simplicity, we assume that $\xi_o^j < 0$. Then, we have

$$\omega(\alpha_0, \beta_o)_1 = \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i} \cdot m_j(i\beta_o) \deg_{\mathcal{V}_{j,1}}. \tag{6.54}$$

In this way, the input data for the Maple© procedure consists of the two sequences:

$$\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}, \quad \{\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r\},$$

where $\mathfrak{t}_j = \mathfrak{t}_{j,1}(\alpha_o, \beta_o)$, $j = 0, 1, \dots, r$. The command for the computation is

$$\omega(\alpha_o, \beta_o)_1 = \texttt{showdegree}\,[\Gamma](\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r, \mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r).$$

In Appendix A4.1, we present a table of computational results for an S_4-symmetric Hopf bifurcation problem in the considered system (6.42), which is listed in a form of a matrix

ξ_o^j	$\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}$	$\omega(\alpha_o, \beta_o)_1$	# Branches

where in the sequence $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\} \subset \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}$, we only list those ε_j which can realize the value 1, and the last colum lists a lower estimate of the number of branches of nonconstant periodic solutions to the system (6.42). More computational examples can be found in [6].

6.4 Global Hopf Bifurcation in Symmetric Functional Differential Equations

In this section, we apply the twisted primary degree method to a global Hopf bifurcation problem in a system of Γ-symmetric functional differential equations, to analyze a continuation of symmetric branches of non-constant periodic solutions.

6.4.1 Abstract Setting

Let $F : \mathbb{R}^2 \oplus W \to W$ be a G-equivariant map satisfying the following assumptions

(H1) F is a compact vector field of class C^1 and $F(\lambda, 0) = 0$ for all $(\lambda, 0) \in \mathbb{R}^2 \oplus W$;

(H2) The set $\Lambda := \{\lambda \in \mathbb{R}^2 : D_w F(\lambda, 0) : W \to W$ is not an isomorphism$\}$ is discrete in $\mathbb{R}^2$;

(H3) $D_{w_o} F|_{\mathbb{R}^2 \oplus W^{S^1}}(\lambda, 0)$ is an isomorphism from W^{S^1} to W^{S^1} for all $\lambda \in \mathbb{R}^2$ and $w_o \in W^{S^1}$.

We are interested in solutions to the equation

$$F(\lambda, w) = 0, \quad (\lambda, w) \in \mathbb{R}^2 \oplus W. \tag{6.55}$$

By (H1), the points $(\lambda, 0)$ are called *trivial solutions* to (6.55). All other solutions will be called *nontrivial*. By implicit function theorem, $(\lambda_o, 0)$ is a bifurcation point only if $\lambda_o \in \Lambda$. By (H2), we obtain that the set of bifurcation points is discrete in $\mathbb{R}^2$.

Let $\mathcal{S}$ be the closure of the set of all nontrivial solutions to (6.55). Notice that $(\lambda_o, 0)$ is a bifurcation point of (6.55) iff $(\lambda_o, 0) \in \mathcal{S}$. Take a connected component $\mathcal{C} \subset \mathcal{S}$. If $\mathcal{C}$ contains a bifurcation point $(\lambda, 0)$, $\mathcal{C}$ is clearly G-invariant. Notice that, in general, $\mathcal{C}$ may be composed of several orbit types, i.e. $\mathcal{C} = \cup_{(H)} \mathcal{C}_{(H)}$, and the global behavior of $\mathcal{C}_{(H)}$ can be different for different orbit types (H), for example, some of the branches $\mathcal{C}_{(H)}$ may be bounded, while the others are unbounded.

The following result can be proved in a standard way and considered as a *global bifurcation theorem*.

Theorem 6.4.1. *Assume* $F : \mathbb{R}^2 \oplus W \to W$ *satisfies the assumptions (H1)—(H3) and let* $\mathcal{C}_{(H_o)}$ *be a bounded connected component of* $\mathcal{S}_{(H_o)}$ *such that* $\mathcal{C}_{(H_o)} \cap \mathbb{R}^2 \times \{0\} = \{(\lambda_1, 0), (\lambda_2, 0), \dots, (\lambda_N, 0)\} \neq \emptyset$, *where* (H_o) *is a dominating orbit type in* W *(cf. Definition 6.1.7). Suppose that* $\omega(\lambda_k) = \sum_{(H)} n_H^k(H)$, *where* $\omega(\lambda_k)$ *are the local* $\Gamma \times S^1$*-invariants around* λ_k. *Then* $\sum_{k=1}^{N} n_{H_o}^k = 0$.

Corollary 6.4.2. *Assume* $F : \mathbb{R}^2 \oplus W \to W$ *satisfies the assumptions (H1) (H3) and let* $\mathcal{C}_{(H_o)}$ *be a connected component of* $\mathcal{S}_{(H_o)}$ *such that* $\mathcal{C}_{(H_o)} \cap \mathbb{R}^2 \times \{0\} = \{(\lambda_1, 0)\}$, *where* (H_o) *is a dominating orbit type in* W. *Suppose that* $\omega(\lambda_1) = \sum_{(H)} n_H^1(H)$, *and* $n_{H_o}^1 \neq 0$. *Then* $\mathcal{C}_{(H_o)}$ *is unbounded.*

6.4.2 Computational Examples

The results obtained above will be applied to a D_N-symmetric and a A_4-symmetric system for the study of the symmetric Hopf bifurcation problems.

Global Hopf Bifurcation in a D_N-Symmetric System

We consider here the system of equations (6.42) with the $N \times N$-matrix C (N an even number) of the type

$$C = \begin{bmatrix} -3 & 1 & 0 \dots 0 & 1 \\ 1 & -3 & 1 \dots 0 & 0 \\ \vdots & \vdots & \vdots \ddots \vdots & \vdots \\ 1 & 0 & 0 \dots 1 & -3 \end{bmatrix}. \tag{6.56}$$

This system is symmetric with respect to the dihedral group $\Gamma = D_N$ acting on $V = \mathbb{R}^N$ by permuting the coordinates of vectors.

Theorem 6.4.3. *(i) Consider system (6.42) with* C *given by (6.56) and suppose* $\eta := h(0)g'(0) > 1$. *Assume:*

(A1) $\dfrac{tg(t)}{h(t)} > 0$ *for all* $t \neq 0$; $\lim_{t \to \infty} \dfrac{tg(t)}{h(t)} = \infty$.

Then the branch $\mathcal{C}_{(D_N^d)}$ *of periodic solutions bifurcating from* $(\alpha_{\frac{n}{2}}, \beta_{\frac{n}{2}}, 0)$ *is unbounded in* $\mathbb{R}^2 \oplus W$.

(ii) Assume, in addition, the following condition is satisfied:

(A2) There exist constants A, $B > 0$ and δ, $\gamma > 0$ with $1 > \delta + \gamma$ such that

$$|h(t)| \leq A + B|t|^{\delta}, \quad |g(t)| \leq A + B|t|^{\gamma}. \tag{6.57}$$

Then,

$$[\alpha_{n/2}, \infty) \subset \left\{\alpha : (\alpha, \beta, x) \in \mathcal{C}_{(D_N^d)}\right\}.$$

Proof: (i) Suppose that (α, β, x) is a solution to (6.42) belonging to $\mathcal{C}_{(D_N^d)}$. Recall that

$$D_N^d = \{(1,1), (\gamma, -1), \dots, (\gamma^{n-1}, -1), (\kappa, 1), (\kappa\gamma, -1), \dots, (\kappa\gamma^{n-1}, -1)\},$$

where γ is 2×2 matrix representing the complex multiplication by $e^{\frac{2\pi i}{n}}$ and $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the operator of complex conjugation. Then, the symmetry properties of $x(t)$ can be translated as follows: $x(t) := \begin{bmatrix} x^0(t) \\ x^1(t) \\ \vdots \\ x^{n-1}(t) \end{bmatrix}$ is a $\frac{2\pi}{\beta}$-periodic solution such that

$$x^k(t) = x^{k+1}\left(t - \frac{\pi}{\beta}\right) \pmod{n}, \tag{6.58}$$

and

$$x^k(t) = x^{n-k-1}\left(t - \frac{\pi}{\beta}\right) \pmod{n}. \tag{6.59}$$

Combining (6.58), (6.59) with condition (A1) and applying the same argument as in [117], one can easily show that the periods $p = \frac{2\pi}{\beta}$ of solutions $(\alpha, \beta, x) \in \mathcal{C}_{(D_N^d)}$ satisfy the inequality $2 < p < 4$. This fact immediately implies $\mathcal{C}^{(D_n^d)} \cap \mathbb{R}^2 \times \{0\} = (\alpha_{\frac{n}{2}+1}, \beta\alpha_{\frac{n}{2}+1}, 0)$ and $\omega(\alpha_{\frac{n}{2}+1}, \beta\alpha_{\frac{n}{2}+1}) = \omega(\alpha_{\frac{n}{2}+1}, \beta\alpha_{\frac{n}{2}+1})_1$.

However, (D_n^d) is a dominating orbit type in W and $\deg_{\mathcal{V}_{\frac{n}{2}+1,1}} = (D_n^d)$, hence $\omega(\alpha_{\frac{n}{2}+1}, \beta\alpha_{\frac{n}{2}+1})_1$ contains a nontrivial coefficient related to (D_n^d), and Corollary 6.4.2 is applied.

(ii) By construction and argument given in (i), $\mathcal{C}_{(D_N^d)} \subset \mathbb{R} \times (\pi/2, \pi) \times W$. Further, using assumption (A2), one can easily show that there exists a constant $M > 0$ such that for every periodic solution $x(t)$ to (6.42) we have

$\sup\{\|x(t)\| : t \in \mathbb{R}\} \leq M$. Indeed, assume that $x(t)$ is a periodic solution of (6.42) and consider the function $r(t) := \|x(t)\|^2$. Since $r(t)$ is periodic, we have that there exists $t_o \in \mathbb{R}$ such that

$$r(t_o) = \sup\{r(t) : t \in \mathbb{R}\}, \quad \text{and} \quad r'(t_o) = 0,$$

i.e. we have

$$\begin{aligned} 0 = \frac{dr}{dt}\Big|_{t=t_o} &= \Big\langle 2x(t_o), x'(t_o)\Big\rangle \\ &= \Big\langle 2x(t_o), -\alpha x(t_o) + \alpha H(x(t_o)) \cdot C(G(x(t_o - 1)))\Big\rangle \\ &= -2\alpha\|x(t_o)\|^2 + \Big\langle 2\alpha x(t_o), H(x(t_o)) \cdot C(G(x(t_o - 1)))\Big\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in V. Therefore, by (A2) we get

$$\begin{aligned} \|x(t_o)\|^2 &\leq \Big|x(t_o) \cdot \Big(H(x(t_o)) \cdot C(G(x(t_o - 1))\Big)\Big| \\ &\leq \|x(t_o)\| \, \|C\| \Big(A + B\|x(t_o)\|^{\delta})\Big)\Big(A + B\|x(t_o + 1)\|^{\gamma})\Big) \\ &\leq c_0 + c_1\|x(t_o)\|^{\delta+1} + c_2\|x(t_o)\|^{\gamma+1} + c_3\|x(t_o)\|^{\delta+\gamma+1}, \end{aligned}$$

for certain constants $c_0, c_1, c_2, c_3 > 0$. Since $\delta + \gamma + 1 < 2$, it follows that there exists a constant $M > 0$ such that every solution s of the inequality

$$s^2 - c_3|s|^{\delta+\gamma+1} - c_2|s|^{\gamma+1} - c_1|s|^{\delta+1} - c_0 \leq 0,$$

satisfies the inequality $|s| \leq M$. Consequently,

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} = \|x(t_o)\| \leq M.$$

Thus, $\mathcal{C}_{(D_N^d)} \subset \mathbb{R} \times (\pi/2; \pi) \times \{x \in W : \|x\| \leq M\}$. Finally, system (6.42) has no non-constant periodic solution for $\alpha = 0$, from which it follows $\mathcal{C}_{(D_N^d)} \subset (0, \infty) \times (\pi/2, \pi) \times \{x \in W : \|x\| \leq M\}$. However, by (i), the connected component $\mathcal{C}_{(D_N^d)}$ is unbounded, therefore $[\alpha_{n/2}, \infty) \subset \{\alpha : (\alpha, \beta, x) \in \mathcal{C}_{(D_N^d)}\}$. □

Global Hopf Bifurcation in a A_4-Symmetric System

We consider here the system of equations (6.42) with the matrix C given by

$$C = \begin{bmatrix} -4 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix}. \tag{6.60}$$

This system is symmetric with respect to the tetrahedral group $\Gamma = A_4$ acting on $V = \mathbb{R}^4$ by permuting the coordinates of vectors. We have $\sigma(C) = \{\xi_0 = -1,\ \xi_1 = -5\}$. The isotypical decomposition of V takes the form: $V = V_0 \oplus V_1$, where V_0 (spanned by the vector $\langle 1,1,1,1\rangle$) is the fixed-point subspace of the A_4-action, and V_1 is equivalent to the natural three-dimensional representation of A_4. These two subspaces are the eigenspaces of the matrix C: the subspace V_0 corresponds to ξ_0 and V_1 to ξ_1. One can verify that the dominating orbit types in W are $(\mathbb{Z}_3^{t_1})$, $(\mathbb{Z}_3^{t_2})$, and (V_4^-). Assuming $\eta > 1$, we are interested in the global behavior of the branch $\mathcal{C}_{(V_4^-)}$ of periodic solutions to (6.42) bifurcating from $(\alpha_1, \beta_1, 0) \in \Lambda \times \{0\}$.

Suppose that (α, β, x) is a solution to (6.42) belonging to $\mathcal{C}_{(V_4^-)}$. Recall that

$$V_4^- = \{((1),1), ((12)(34),1), ((13)(24),-1), ((14)(23),-1)\}.$$

Then the symmetry properties of $x(t)$ can be translated as follows: $x(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \\ x^4(t) \end{bmatrix}$ with

$$x^2(t) = x^1\left(t - \frac{\pi}{\beta}\right), \quad x^4(t) = x^2\left(t - \frac{\pi}{\beta}\right), \tag{6.61}$$

$$x^4(t) = x^1\left(t - \frac{\pi}{\beta}\right), \quad x^3(t) = x^2\left(t - \frac{\pi}{\beta}\right) \tag{6.62}$$

Using (6.61), (6.62) and following the same lines as in the case of dihedral symmetries, one can easily establish

Theorem 6.4.4. *(i) Consider system (6.42) with C given by (6.60) and suppose* $\eta := h(0)g'(0) > 1$. *Assume condition (A1) is satisfied. Then the branch* $\mathcal{C}_{(V_4^-)}$ *of periodic solutions bifurcating from* $(\alpha_1, \beta_1, 0)$ *is unbounded in* $\mathbb{R}^2 \oplus W$.

(ii) Assume, in addition, condition (A2) is satisfied. Then

$$[\alpha_1, \infty) \subset \left\{\alpha : (\alpha, \beta, x) \in \mathcal{C}_{(V_4^-)}\right\}.$$

7

Hopf Bifurcation in Symmetric Systems of Neutral Functional Differential Equations

In this chapter, we present another application of the (twisted) primary equivariant degree method to a Γ-symmetric Hopf bifurcation problem for a system of neutral functional differential equations, motivated by a model of two types of symmetrically coupled configurations of the lossless transmission lines. The standard degree-theoretical treatment, which was introduced in Section 6.1, is adapted to this type of systems. We follow exactly the same steps as in Section 6.1, namely, we inspect the characteristic equation for the occurrence of purely imaginary roots (to identify the isolated centers), analyze the equivariant spectral properties of the characteristic operator to determine the isotypical crossing numbers and multiplicities of the negative eigenvalues (associated to the considered center). Then, the local bifurcation invariant can be computed according to a similar formula as (6.41) (cf. (7.9)). Finally, exact values of the bifurcation invariants can be evaluated with the assistance of the Maple© routines. Computational sample results for the local $\Gamma \times S^1$-invariants can be found in Appendix A4.2, for $\Gamma = D_4, A_5$.

The chapter is organized as follows. In Section 7.1, we state the symmetric Hopf bifurcation problem in a system of neutral functional differential equations and set up a framework for the standard degree-theoretical approach. A local bifurcation invariant is associated to an isolated center and we derive a computational formula (cf. (7.9)). In Section 7.2, we discuss models for two systems of symmetrically coupled (internally and externally) lossless transmission lines, based on the telegrapher's equation. Motivated by the two generic couplings, we consider in Section 7.3, we consider a symmetric system of NFDEs, for which we carry out an analysis for the occurrence of the symmetric Hopf bifurcation. Th concrete computational results for $\Gamma = D_4, A_5$ are summarized in Appendix A4.2.

7.1 Hopf Bifurcation in Symmetric Systems of NFDEs

Throughout this chapter, we assume that $G = \Gamma \times S^1$, where Γ is a finite group.

Suppose that V is a Γ-orthogonal representation. For a given constant $\tau \geq 0$, let $C_{V,\tau}$ be an isometric Banach Γ-representation defined by (6.1)–(6.3). We consider an $\mathbb{R}$-parametrized system of neutral functional differential equations

$$\frac{d}{dt}\Big[x(t) - b(\alpha, x_t)\Big] = f(\alpha, x_t), \tag{7.1}$$

where $x : \mathbb{R} \to V$ is a continuous function[*] , $x_t \in C_{V,\tau}$ is defined by (6.4), and $b, f : \mathbb{R} \oplus C_{V,\tau} \to V$ satisfy the following assumptions

(A1) b, f are continuously differentiable;
(A2) b, f are Γ-equivariant;
(A3) $b(\alpha, 0) = 0$, $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.

Also, to prevent the occurrence of the steady-state bifurcation, assume

(A4) $\det D_x f(\alpha, 0)|_V \neq 0$ for all $\alpha \in \mathbb{R}$.

In addition, assume that

(A5) b satisfies the Lipschitz condition with respect to the second variable, i.e.

$$\exists_\kappa \ 0 \leq \kappa < 1, \text{ s.t. } \|b(\alpha, \varphi) - b(\alpha, \psi)\| \leq \kappa \|\varphi - \psi\|_\infty \tag{7.2}$$

for all $\varphi, \psi \in C_{V,\tau}$, $\alpha \in \mathbb{R}$.

Similar as in Section 6.1, we call $(\alpha, x_o) \in \mathbb{R} \oplus V$ a *stationary* point to (7.1), if $f(\alpha, x_o) = 0$. By assumption (A3), $(\alpha, 0)$ is a stationary point for all $\alpha \in \mathbb{R}$. A stationary point (α, x_o) is said to be *nonsingular* if $D_x f(\alpha, x_o) : V \to V$ is a linear isomorphism.

7.1.1 Characteristic Equation

Let (α, x_o) be a stationary point of (7.1). The linearization of (7.1) at (α, x_o) leads to the characteristic equation

$$\det{}_{\mathbb{C}} \Delta_{(\alpha, x_o)}(\lambda) = 0, \tag{7.3}$$

---

[*] Formally speaking, we only need to require $x(t) - b(\alpha, x_t)$ to be continuously differentiable.

where

$$\triangle_{(\alpha,x_o)}(\lambda) := \lambda\Big[\mathrm{Id} - D_x b(\alpha, x_o)(e^{\lambda\cdot}\cdot)\Big] - D_x f(\alpha, x_o)(e^{\lambda\cdot}\cdot) \tag{7.4}$$

is a complex linear operator from V^c to V^c.

Similar definitions of characteristic roots, centers and isolated centers will be adopted from Section 6.1. The same notations used in Subsection 6.1.1—6.1.2 concerning the characteristic operator and the isotypical decompositions will be kept without further notice.

We will assume additionally that

(A6) The system (7.1) has an isolated center $(\alpha_o, 0)$ for some $\alpha_o \in \mathbb{R}$, with the corresponding purely imaginary characteristic root $i\beta_o$, for $\beta_o > 0$.

Our interesting problem is to study the Γ-symmetric Hopf bifurcation problem in the system (7.1) around an isolated center $(\alpha_o, 0)$, including the detection of nonconstant periodic solutions and the symmetric classification of the solution set according to different subsymmetries. We will follow the similar procedure described in Subsection 6.1.3— 6.1.5 and associate a local bifurcation invariant in terms of a twisted $\Gamma \times S^1$-primary equivariant degree, to the system (7.1) at the isolated center $(\alpha_o, 0)$.

7.1.2 Normalization of Period

We transform the problem of finding a p-periodic solution to a problem of finding a 2π-periodic solution by making the change of variable $x(t) = u(\beta t)$, where $\beta := \frac{2\pi}{p}$ is an additional parameter. Then, from the system (7.1), we obtain the following

$$\frac{d}{dt}\Big[u(t) - b(\alpha, u_{t,\beta})\Big] = \frac{1}{\beta} f(\alpha, u_{t,\beta}), \tag{7.5}$$

where $u_{t,\beta} \in C_{V,\tau}$ is defined by (6.14). Evidently, $u(t)$ is a 2π-periodic solution of (7.5) if and only if $x(t)$ is a p-periodic solution of (7.1).

7.1.3 Setting in Functional Spaces

We use the standard identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and define $W := H^1(S^1; V)$, which is naturally an isometric Hilbert representation of G (cf. (6.21)).

Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$. Let the operations L, j, K and N_f be given by (6.16)—(6.18) and (6.20) respectively. For $u \in W$, $v \in C(S^1;V)$, $t \in \mathbb{R}$, define $N_b : \mathbb{R}^2_+ \times C(S^1;V) \to L^2(S^1;V)$ by

$$N_b(\alpha,\beta,v)(t) = b(\alpha, v_{t,\beta}).$$

Moreover, define the map $\mathcal{F} : \mathbb{R}^2_+ \times W \to W$ by

$$\mathcal{F}(\alpha,\beta,u) = (L+K)^{-1}\left[\frac{1}{\beta}N_f(\alpha,\beta,u)+K(u-N_b(\alpha,\beta,u))\right]+N_b(\alpha,\beta,u), \quad (7.6)$$

which is a condensing map. Indeed, the map $\mathcal{F}$ is a sum of two maps, where the first map

$$(\alpha,\beta,u) \mapsto (L+K)^{-1}\left[\frac{1}{\beta}N_f(\alpha,\beta,u) + K(u - N_b(\alpha,\beta,u))\right],$$

is completely continuous, and the second map $(\alpha,\beta,u) \mapsto N_b(\alpha,\beta,u)$ is a Banach contraction with constant κ $(0 \le \kappa < 1)$ (cf. (A1) and (A5)).

7.1.4 Sufficient Condition for Symmetric Hopf Bifurcation

Following the same construction outlined in Subsection 6.1.5, we define a region $\Omega \subset \mathbb{R}^2_+ \times W$ by (6.22), an auxiliary function ς by (6.23) and a map $\mathfrak{F}_\varsigma$ by (6.24), which is clearly an Ω-admissible G-equivariant condensing field (cf. Section 2.7). By the standard Nussbaum-Sadovskii extension, one can define the equivariant degree theory to equivariant condensing fields on Hilbert isometric G-representations (cf. [15] for more details). We use the same symbol to denote this extended equivariant degree.

Definition 7.1.1. Let $\Omega, \varsigma, \mathfrak{F}_\varsigma$ be defined by (6.22), (6.23) and (6.24) respectively. We call

$$\omega(\lambda_o) := G\text{-Deg}(\mathfrak{F}_\varsigma, \Omega) \in A_1(G), \quad (7.7)$$

the *local G-invariant* for the Γ-symmetric Hopf bifurcation of the system (7.1) at $(\lambda_o, 0)$.

Similarly to Theorem 6.1.9, we have the following result for the symmetric Hopf bifurcation problem in (7.1).

Theorem 7.1.2. *Given system (7.1), assume conditions (A1)—(A6) to be satisfied. Let $\mathcal{F}$ be defined by (7.6) and Ω, ς, $\mathfrak{F}_\varsigma$ given by (6.22)—(6.24) respectively.*

(i) *Assume* $\omega(\lambda_o) \neq 0$ *(cf. (7.7)), i.e.*

$$\omega(\lambda_o) = \sum_{(H)} n_H(H) \quad \text{and} \quad n_{H_o} \neq 0 \tag{7.8}$$

for some $(H_o) \in \Phi_1(G)$. *Then, there exists a branch of non-trivial solutions to (7.1) bifurcating from the point* $(\alpha_o, 0)$ *(with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$*). More precisely, the closure of the set composed of all non-trivial solutions* $(\lambda, u) \in \Omega$ *to (7.5), i.e.*

$$\overline{\{(\lambda, u) \in \Omega : \ \mathfrak{F}(\lambda, u) = 0, \ u \neq 0\}}$$

contains a compact connected subset C *such that*

$$(\lambda_o, 0) \in C \quad \text{and} \quad C \cap \partial_r \neq \emptyset, \quad C \subset \mathbb{R}^2_+ \times W^{H_o},$$

$(\lambda_o = \alpha_o + i\beta_o)$ *which, in particular, implies that for every* $(\alpha, \beta, u) \in C$ *we have* $G_u \supset H_o$.

(ii) *If, in addition,* (H_o) *is a dominating orbit type in* W, *then there exist at least* $|G/H_o|_{S^1}$ *different branches of periodic solutions to the equation (7.1) bifurcating from* $(\alpha_o, 0)$ *(with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$*). Moreover, for each* (α, β, u) *belonging to these branches of (non-trivial) solutions one has* $(G_u) = (H_o)$ *(considered in the space* W*).*

7.1.5 Computational Formula for the Local Invariant

To apply Theorem 7.1.2, we need to establish an effective computational formula for $\omega(\lambda_o)$. Notice that the linearization procedure and the reduction through isotypical decompositions discussed in Section 6.2, do not wear specific restrictions from the functional setting and thus apply effectively to the current setting.

Therefore, we have the following computational formula for $\omega(\lambda_o)$ (cf. (6.41))

$$\omega(\lambda_o) = \prod_{\mu \in \sigma_-(\overline{A})} \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)} \cdot \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o, \beta_o) \deg_{\mathcal{V}_{j,l}}, \tag{7.9}$$

where $m_i(\mu)$ is the $\mathcal{V}_i$-multiplicity of μ (cf. (4.4)), $\mathfrak{t}_{j,l}$ are the isotypical crossing numbers (cf. (6.10)–(6.12)) and $\deg_{\mathcal{V}_i}$, $\deg_{\mathcal{V}_{j,l}}$ are the basic degrees (cf. Subsection 4.1.3 and 4.2.4).

7.2 Symmetric Configurations of Lossless Transmission Line Models

In this section, we consider two simple generic types of symmetric configurations for the lossless transmission line models, and derive symmetric systems of neutral functional differential equations, which give insight of reasonable symmetries one could expect in such models.

7.2.1 Configuration 1: Internal Coupling

Consider first a cube of symmetrically coupled lossless transmission line networks between two recipients (C) and two power stations (E). Assume all coupled networks are identical, each of which is a uniformly distributed lossless transmission line with the inductance L_s and parallel capacitance C_s per unit length. To derive the network equations, we place the x-axis in the direction of each line, with two ends of the normalized line at $x = 0$ and $x = 1$ (cf. Figure 7.1)*.

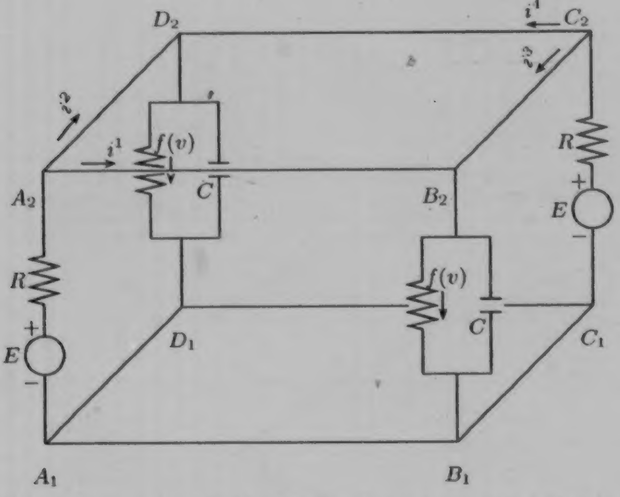

Fig. 7.1. Symmetric Model of Transmission Lines: Internal Coupling

Denote by $i^j(x,t)$ the current flowing in the j-th line at time t and distance x down the line and $v^j(x,t)$ the voltage across the line at t and x, for $j = 1, 2, 3, 4$.

---

[*] This example of internal coupling can be easily generalized for a coupling of N recipients and N power stations with an $N > 2$.

It is well-known that (see, for instance, [129]) the functions $i^j := i^j(x,t)$ and $v^j := v^j(x,t)$ obey the following partial differential equations (*Telegrapher's equation*)

$$\begin{cases} \frac{\partial v^j}{\partial x} = -L_s \frac{\partial i^j}{\partial t}, \\ C_s \frac{\partial v^j}{\partial t} = -\frac{\partial i^j}{\partial x}. \end{cases} \tag{7.10}$$

When these networks are coupled symmetrically in the way shown in Figure 7.1, the vertical lines have coupling terms from the preceding and succeeding lines at each end $x = 0$ and $x = 1$, thus it gives rise to the boundary conditions

$$\begin{cases} E = v_0^1 + (i_0^1 + i_0^2)R, \\ i_1^1 + i_1^3 = f(v_1^1) + C\frac{dv_1^1}{dt}, \\ E = v_0^3 + (i_0^3 + i_0^4)R, \\ i_1^2 + i_1^4 = f(v_1^2) + C\frac{dv_1^2}{dt}, \\ v_0^1 = v_0^2, \quad v_0^3 = v_0^4, \\ v_1^1 = v_1^3, \quad v_1^2 = v_1^4, \end{cases} \tag{7.11}$$

where $i_\delta^j = i_\delta^j(t) := i^j(\delta, t)$, $v_\delta^j = v_\delta^j(t) := v^j(\delta, t)$ for $\delta \in \{0, 1\}$, E is the constant direct current voltage and $f(v_1^j)$ is the current through the nonlinear resistor in the direction shown in Figure 7.1.

For mathematical simplicity, we assume that

(E1) the boundary value problem (7.10)-(7.11) admits a unique solution $(v_*^j, i_*^j) := (v_*^j(x,t), i_*^j(x,t))$, for $j = 1,2,3,4$ such that $\frac{\partial i_*^j}{\partial x} = \frac{\partial v_*^j}{\partial x} = 0$ (the so-called *equilibrium point*).

Thus, the equilibrium point (v_*^j, i_*^j), $j = 1,2,3,4$ satisfies the following *equilibrium equations*:

$$\begin{cases} E = v_*^1 + (i_*^1 + i_*^2)R, \\ i_*^1 + i_*^3 = f(v_*^1) + C\frac{dv_*^1}{dt}, \\ E = v_*^3 + (i_*^3 + i_*^4)R, \\ i_*^2 + i_*^4 = f(v_*^2) + C\frac{dv_*^2}{dt}. \end{cases} \tag{7.12}$$

Now, subtract the first four equations in (7.11) by (7.12), we obtain

$$\begin{cases} 0 = v_0^1 - v_*^1 + (i_0^1 - i_*^1 + i_0^2 - i_*^2)R, \\ i_1^1 - i_*^1 + i_1^3 - i_*^3 = f(v_1^1) - f(v_*^1) + C\frac{d}{dt}(v_1^1 - v_*^1), \\ 0 = v_0^3 - v_*^3 + (i_0^3 - i_*^3 + i_0^4 - i_*^4)R, \\ i_1^2 - i_*^2 + i_1^4 - i_*^4 = f(v_1^2) - f(v_*^2) + C\frac{d}{dt}(v_1^2 - v_*^2). \end{cases} \tag{7.13}$$

By changing variables, letting $\mathcal{X}_\delta^j = v_\delta^j - v_*^j$, $\mathcal{Y}_\delta^j = i_\delta^j - i_*^j$ (for $\delta = 0, 1$) and setting

$$g(\mathcal{X}_1^j) := f(\mathcal{X}_1^j + v_*^j) - f(v_*^j) = f(v_1^j) - f(v_*^j), \tag{7.14}$$

we have the boundary conditions (7.11) reduce to

$$\begin{cases} 0 = \mathcal{X}_0^1 + (\mathcal{Y}_0^1 + \mathcal{Y}_0^2)R, \\ \mathcal{Y}_1^1 + \mathcal{Y}_1^3 = g(\mathcal{X}_1^1) + C\frac{d\mathcal{X}_1^1}{dt}, \\ 0 = \mathcal{X}_0^3 + (\mathcal{Y}_0^3 + \mathcal{Y}_0^4)R, \\ \mathcal{Y}_1^2 + \mathcal{Y}_1^4 = g(\mathcal{X}_1^2) + C\frac{d\mathcal{X}_1^2}{dt}, \\ \mathcal{X}_0^1 = \mathcal{X}_0^2, \quad \mathcal{X}_0^3 = \mathcal{X}_0^4, \\ \mathcal{X}_1^1 = \mathcal{X}_1^3, \quad \mathcal{X}_1^2 = \mathcal{X}_1^4. \end{cases}$$

For simplicity, we replace the symbols $\mathcal{X}_\delta^j$ and $\mathcal{Y}_\delta^j$ with v_δ^j and i_δ^j respectively (for $\delta = 0, 1$),

$$\begin{cases} 0 = v_0^1 + (i_0^1 + i_0^2)R, \\ i_1^1 + i_1^3 = g(v_1^1) + C\frac{dv_1^1}{dt}, \\ 0 = v_0^3 + (i_0^3 + i_0^4)R, \\ i_1^2 + i_1^4 = g(v_1^2) + C\frac{dv_1^2}{dt}, \\ v_0^1 = v_0^2, \quad v_0^3 = v_0^4, \\ v_1^1 = v_1^3, \quad v_1^2 = v_1^4. \end{cases} \tag{7.15}$$

Our goal is to reduce the boundary value problem (7.10) and (7.15) to a system of symmetric NFDEs. To this end, recall that the general solution to (7.10) (the so-called *d'Alembert solution*) takes the form:

$$\begin{cases} v^j(x,t) &= \frac{1}{2}[\phi^j(x - at) + \psi^j(x + at)], \\ i^j(x,t) &= \frac{1}{2b}[\phi^j(x - at) - \psi^j(x + at)], \end{cases} \tag{7.16}$$

where

$$a = \frac{1}{\sqrt{L_s C_s}}, \quad b = \sqrt{\frac{L_s}{C_s}} \tag{7.17}$$

are respectively the propagation velocity of waves and the characteristic impedance of the line, and $\phi^j \in C^1((-\infty, 1]; \mathbb{R})$, $\psi^j \in C^1([0, \infty); \mathbb{R})$ (see, for instance, [169]).

Next, we will essentially use the identity

$$i^j(x,t) + i^j\left(x, t - \frac{2}{a}\right) = i^j\left(x - 1, t - \frac{1}{a}\right) + i^j\left(x + 1, t - \frac{1}{a}\right), \tag{7.18}$$

supported by the following verification

$$\begin{aligned}
i^j(x,t) &= \frac{1}{2b}[\phi^j(x - at) - \psi^j(x + at)] \\
&= \frac{1}{2b}\left[\phi^j\left(x - 1 - a(t - \frac{1}{a})\right) - \psi^j\left(x + 1 + a(t - \frac{1}{a})\right)\right] \\
&= \frac{1}{2b}\left[\phi^j\left(x - 1 - a(t - \frac{1}{a})\right) - \psi^j\left(x - 1 + a(t - \frac{1}{a})\right)\right] \\
&+ \frac{1}{2b}\left[\psi^j\left(x - 1 + a(t - \frac{1}{a})\right) - \phi^j\left(x + 1 - a(t - \frac{1}{a})\right)\right] \\
&+ \frac{1}{2b}\left[\phi^j\left(x + 1 - a(t - \frac{1}{a})\right) - \psi^j\left(x + 1 + a(t - \frac{1}{a})\right)\right] \\
&= \frac{1}{2b}\left[\phi^j\left(x - 1 - a(t - \frac{1}{a})\right) - \psi^j\left(x - 1 + a(t - \frac{1}{a})\right)\right] \\
&- \frac{1}{2b}\left[\phi^j\left(x - a(t - \frac{2}{a})\right) - \psi^j\left(x + a(t - \frac{2}{a})\right)\right] \\
&+ \frac{1}{2b}\left[\phi^j\left(x + 1 - a(t - \frac{1}{a})\right) - \psi^j\left(x + 1 + a(t - \frac{1}{a})\right)\right] \\
&= i^j\left(x - 1, t - \frac{1}{a}\right) - i^j\left(x, t - \frac{2}{a}\right) + i^j\left(x + 1, t - \frac{1}{a}\right).
\end{aligned}$$

In particular, by substituting $x = 1$ in (7.18), we have

$$i^j\left(2, t - \frac{1}{a}\right) = i_1^j(t) + i_1^j\left(t - \frac{2}{a}\right) - i_0^j\left(t - \frac{1}{a}\right). \tag{7.19}$$

Return to the boundary conditions (7.15). Using (7.16), we obtain:

$$\begin{cases}
\phi^1(-at) = \frac{R-b}{R+b}\psi^1(at) - \frac{2bR}{R+b}i_0^2(t), \\
\phi^3(-at) = \frac{R-b}{R+b}\psi^3(at) - \frac{2bR}{R+b}i_0^4(t).
\end{cases}$$

Consequently,

$$\begin{aligned}
C\frac{dv_1^1}{dt} &= i_1^1 + i_1^3 - g(v_1^1) \\
&= \frac{\phi^1(1 - at) - v_1^1}{b} + \frac{\phi^3(1 - at) - v_1^3}{b} - g(v_1^1) \\
&= \frac{\phi^1(-a(t - \frac{1}{a})) - v_1^1}{b} + \frac{\phi^3(-a(t - \frac{1}{a})) - v_1^3}{b} - g(v_1^1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{R-b}{R+b}\psi^1(at-1) - \frac{2bR}{R+b}i_0^2(t-\frac{1}{a}) - v_1^1}{b} \\
&+ \frac{\frac{R-b}{R+b}\psi^3(at-1) - \frac{2bR}{R+b}i_0^4(t-\frac{1}{a}) - v_1^3}{b} - g(v_1^1) \qquad (7.20)
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
&C\frac{R-b}{R+b}\frac{dv_1^1}{dt}(t-\frac{2}{a}) \\
&= \frac{R-b}{R+b}[i_1^1(t-\frac{2}{a}) + i_1^3(t-\frac{2}{a}) - g(v_1^1(t-\frac{2}{a}))] \\
&= \frac{R-b}{R+b}[\frac{v_1^1(t-\frac{2}{a}) - \psi^1(1+a(t-\frac{2}{a}))}{b} \\
&+ \frac{v_1^3(t-\frac{2}{a}) - \psi^3(1+a(t-\frac{2}{a}))}{b} - g(v_1^1(t-\frac{2}{a}))] \\
&= \frac{1}{b}\frac{R-b}{R+b}[v_1^1(t-\frac{2}{a}) + v_1^3(t-\frac{2}{a})] \\
&- \frac{1}{b}\frac{R-b}{R+b}[\psi^1(at-1) + \psi^3(at-1)] - \frac{R-b}{R+b}g(v_1^1(t-\frac{2}{a})). \qquad (7.21)
\end{aligned}
$$

Combining (7.20) and (7.21) results in

$$
\begin{aligned}
&C[\frac{dv_1^1}{dt} + \frac{R-b}{R+b}\frac{dv_1^1}{dt}(t-\frac{2}{a})] \\
&= -\frac{2R}{R+b}[i_0^2(t-\frac{1}{a}) + i_0^4(t-\frac{1}{a})] - \frac{1}{b}(v_1^1 + v_1^3) \\
&+ \frac{1}{b}\frac{R-b}{R+b}[v_1^1(t-\frac{2}{a}) + v_1^3(t-\frac{2}{a})] - g(v_1^1) - \frac{R-b}{R+b}g(v_1^1(t-\frac{2}{a})). \qquad (7.22)
\end{aligned}
$$

On the other hand, since by (7.16),

$$
\begin{aligned}
i_0^2(t-\frac{1}{a}) &= \frac{1}{2b}[\phi^2(1-at) - \psi^2(at-1)] \\
&= \frac{1}{2b}[2v_1^2 - \psi^2(1+at) - \psi^2(at-1)] \\
&= \frac{1}{b}v_1^2 - \frac{1}{2b}[\psi^2(1+at) + \psi^2(at-1)] \\
&= \frac{1}{b}v_1^2 - \frac{1}{2b}[\psi^2(1+at) + 2v_1^2(t-\frac{2}{a}) - \phi^2(3-at)] \\
&= \frac{1}{b}v_1^2 - \frac{1}{b}v_1^2(t-\frac{2}{a}) + \frac{1}{2b}[\phi^2(3-at) - \psi^2(1+at)] \\
&= \frac{1}{b}v_1^2 - \frac{1}{b}v_1^2(t-\frac{2}{a}) + i^2(2, t-\frac{1}{a}),
\end{aligned}
$$

it follows from (7.19) that

$$i_0^2(t-\frac{1}{a}) = \frac{1}{2b}v_1^2 - \frac{1}{2b}v_1^2(t-\frac{2}{a}) + \frac{1}{2}[i_1^2(t) + i_1^2(t-\frac{2}{a})]. \tag{7.23}$$

Symmetrically, a similar statement is valid for i_0^4, i.e.

$$i_0^4(t-\frac{1}{a}) = \frac{1}{2b}v_1^4 - \frac{1}{2b}v_1^4(t-\frac{2}{a}) + \frac{1}{2}[i_1^4(t) + i_1^4(t-\frac{2}{a})]. \tag{7.24}$$

Using the boundary conditions (7.15) and (7.23) (7.24), we have

$$\begin{aligned}
&i_0^2(t-\frac{1}{a}) + i_0^4(t-\frac{1}{a})\\
&= \frac{1}{b}v_1^2 - \frac{1}{b}v_1^2(t-\frac{2}{a})\\
&+ \frac{1}{2}[g(v_1^2) + C\frac{dv_1^2}{dt} + g(v_1^2(t-\frac{2}{a})) + C\frac{dv_1^2}{dt}(t-\frac{2}{a})].
\end{aligned} \tag{7.25}$$

Therefore, by substituting (7.25) into (7.22) and using the last equality from (7.15), we obtain:

$$\begin{aligned}
&C[\frac{dv_1^1}{dt} + \frac{R-b}{R+b}\frac{dv_1^1}{dt}(t-\frac{2}{a})]\\
&= -\frac{2R}{R+b}[\frac{1}{b}v_1^2 - \frac{1}{b}v_1^2(t-\frac{2}{a})] - \frac{R}{R+b}[g(v_1^2) + C\frac{dv_1^2}{dt}\\
&+ g(v_1^2(t-\frac{2}{a})) + C\frac{dv_1^2}{dt}(t-\frac{2}{a})] - \frac{2}{b}v_1^1 + \frac{2}{b}\frac{R-b}{R+b}v_1^1(t-\frac{2}{a})\\
&= -g(v_1^1) - \frac{R-b}{R+b}g(v_1^1(t-\frac{2}{a})) - C\frac{R}{R+b}[\frac{dv_1^2}{dt} + \frac{dv_1^2}{dt}(t-\frac{2}{a})]\\
&- \frac{2}{b}v_1^1 + \frac{2}{b}\frac{R-b}{R+b}v_1^1(t-\frac{2}{a}) - g(v_1^1) - \frac{R-b}{R+b}g(v_1^1(t-\frac{2}{a}))\\
&- \frac{2R}{R+b}[\frac{1}{b}v_1^2 - \frac{1}{b}v_1^2(t-\frac{2}{a})] - \frac{R}{R+b}[g(v_1^2) + g(v_1^2(t-\frac{2}{a}))],
\end{aligned}$$

which, after rearrangement, yields

$$\begin{aligned}
&C[\frac{dv_1^1}{dt} + \frac{R}{R+b}\frac{dv_1^2}{dt} + \frac{R-b}{R+b}\frac{dv_1^1}{dt}(t-\frac{2}{a}) + \frac{R}{R+b}\frac{dv_1^2}{dt}(t-\frac{2}{a})]\\
&= -\frac{2}{b}v_1^1 + \frac{2}{b}\frac{R-b}{R+b}v_1^1(t-\frac{2}{a}) - \frac{2}{b}\frac{R}{R+b}[v_1^2 - v_1^2(t-\frac{2}{a})]\\
&- g(v_1^1) - \frac{R-b}{R+b}g(v_1^1(t-\frac{2}{a})) - \frac{R}{R+b}[g(v_1^2) + g(v_1^2(t-\frac{2}{a}))].
\end{aligned} \tag{7.26}$$

By the same argument, we obtain:

$$\begin{aligned}
&C\Big[\frac{dv_1^2}{dt} + \frac{R}{R+b}\frac{dv_1^1}{dt} + \frac{R-b}{R+b}\frac{dv_1^2}{dt}(t-\frac{2}{a}) + \frac{R}{R+b}\frac{dv_1^1}{dt}(t-\frac{2}{a})\Big] \\
&= -\frac{2}{b}v_1^2 + \frac{2}{b}\frac{R-b}{R+b}v_1^2(t-\frac{2}{a}) - \frac{2}{b}\frac{R}{R+b}[v_1^1 - v_1^1(t-\frac{2}{a})] \\
&\quad - g(v_1^2) - \frac{R-b}{R+b}g(v_1^2(t-\frac{2}{a})) - \frac{R}{R+b}[g(v_1^1) + g(v_1^1(t-\frac{2}{a}))]. \qquad (7.27)
\end{aligned}$$

In terms of matrices, the system (7.26)–(7.27) can be rewritten as

$$\begin{aligned}
&C\left[S_1\frac{d}{dt}x(t) - S_2\frac{d}{dt}x(t-r)\right] \\
&= -S_3x(t) - S_4x(t-r) - S_5G(x(t)) + S_6G(x(t-r)), \qquad (7.28)
\end{aligned}$$

where

$$r = \frac{2}{a}, \quad x(t) = \begin{bmatrix} v_1^1(t) \\ v_1^2(t) \end{bmatrix}, \quad G(x(t)) = \begin{bmatrix} g(v_1^1(t)) \\ g(v_1^2(t)) \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & \frac{R}{R+b} \\ \frac{R}{R+b} & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} \frac{b-R}{R+b} & -\frac{R}{R+b} \\ -\frac{R}{R+b} & \frac{b-R}{R+b} \end{bmatrix},$$

$$S_3 = \begin{bmatrix} \frac{2}{b} & \frac{2}{b}\frac{R}{R+b} \\ \frac{2}{b}\frac{R}{R+b} & \frac{2}{b} \end{bmatrix} = \frac{2}{b}S_1, \quad S_4 = \begin{bmatrix} -\frac{2}{b}\frac{R-b}{R+b} & -\frac{2}{b}\frac{R}{R+b} \\ -\frac{2}{b}\frac{R}{R+b} & -\frac{2}{b}\frac{R-b}{R+b} \end{bmatrix} = \frac{2}{b}S_2,$$

$$S_5 = \begin{bmatrix} 1 & \frac{R}{R+b} \\ \frac{R}{R+b} & 1 \end{bmatrix} = S_1, \quad S_6 = \begin{bmatrix} \frac{b-R}{R+b} & -\frac{R}{R+b} \\ -\frac{R}{R+b} & \frac{b-R}{R+b} \end{bmatrix} = S_2.$$

Multiplying (7.28) by S_1^{-1} (recall that $b \neq 0$ (see (7.17))), we arrive at

$$\begin{aligned}
&\frac{d}{dt}[x(t) - Qx(t-r)] \\
&= -\frac{2}{bC}x(t) - \frac{2}{bC}Qx(t-r) - \frac{1}{C}G(x(t)) + \frac{1}{C}QG(x(t-r)), \qquad (7.29)
\end{aligned}$$

where $Q = S_1^{-1}S_2$.

Notice that the system (7.28) embodies the symmetric situation, namely the internal coupling, in the following way: let $\Gamma := D_2$ act on $V := \mathbb{R}^2$ by permuting the coordinates of vectors $x = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \in V$, then the system (7.28) is symmetric with respect to the Γ-action on V.

7.2.2 Configuration 2: External Coupling

A second example of symmetric coupling was considered in [180] , where N recipients are mutually coupled via lossless transmission line network which are interconnected by a common resistor R_o between neighboring recipients, and extensively connected with N power stations.

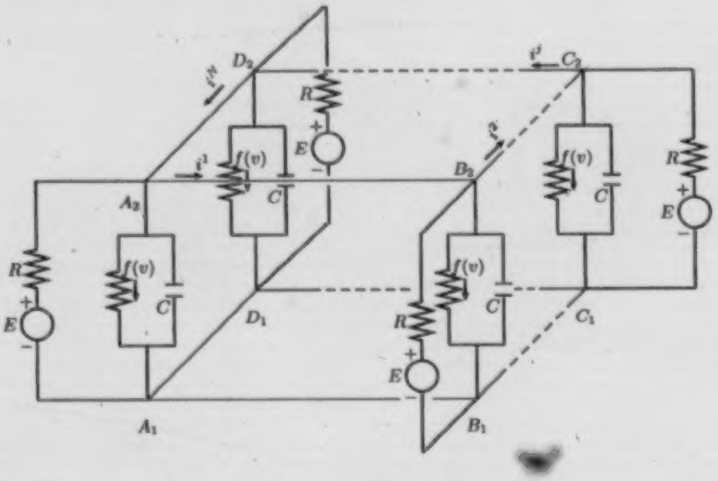

Fig. 7.2. Symmetric Model of Transmission Lines: External Coupling

Denote by $i^j(x,t)$ the current flowing in the j-th line at time t and distance x down the line and $v^j(x,t)$ the voltage across the line at t and x, for $j = 1, \dots, N$. The same Telegrapher's equation (7.10) holds for $i^j(x,t)$ and $v^j(x,t)$. However, the boundary conditions need to be modified for this external coupling. For $j = 1, \cdots, N$, we have

$$\begin{cases} E = v_0^j + i_0^j R, \\ i_1^j = f(v_1^j) + C\frac{dv_1^j}{dt} - (I^{j-1}(t) - I^j(t)), \\ v_1^j - v_1^{j+1} = I^j(t)R_o, \end{cases} \tag{7.30}$$

where $I^0(t) := I^N(t)$, $v^{N+1} := v^1$, I^j's are the so-called coupling terms (see [180]).

For mathematical simplicity, we assume that (cf. (E1))

(E2) the boundary value problem (7.10) and (7.30) admits a unique equilibrium point $(v_*^j, i_*^j) := (v_*^j(x,t), i_*^j(x,t))$, for $j = 1, \cdots, N$.

By a change of variables provided by (7.14), the boundary conditions (7.30) can be translated to

$$\begin{cases} 0 = v_0^j + i_0^j R, \\ i_1^j = g(v_1^j) + C\frac{dv_1^j}{dt} - \frac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}). \end{cases} \tag{7.31}$$

We are now in a position to reduce the boundary value problem (7.10) and (7.31) to a symmetric system of FDEs. By (7.31) and (7.16), we have

$$\phi^j(-at) = \frac{R-b}{R+b}\psi^j(at),$$

and

$$\begin{aligned} C\frac{dv_1^j}{dt} &= i_1^j - g(v_1^j) + \frac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}) \\ &= \frac{\phi^j(1-at) - v_1^j}{b} - g(v_1^j) + \frac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}) \\ &= \frac{\phi^j(-a(t-\frac{1}{a})) - v_1^j}{b} - g(v_1^j) + \frac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}) \\ &= \frac{\frac{R-b}{R+b}\psi^j(at-1) - v_1^j}{b} - g(v_1^j) + \frac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}). \end{aligned}$$

Similarly, we get

$$\begin{aligned} & C\frac{R-b}{R+b}\frac{dv_1^j}{dt}(t-\frac{2}{a}) \\ &= \frac{R-b}{R+b}i_1^j(t-\frac{2}{a}) - \frac{R-b}{R+b}g(v_1^j(t-\frac{2}{a})) \\ &+ \frac{1}{R_o}\frac{R-b}{R+b}(v_1^{j+1} - 2v_1^j + v_1^{j-1})\frac{R-b}{R+b}\frac{v_1^j(t-\frac{2}{a}) - \psi^j(at-1))}{b} \\ &= -\frac{R-b}{R+b}g(v_1^j(t-\frac{2}{a})) + \frac{1}{R_o}\frac{R-b}{R+b}(v_1^{j+1} - 2v_1^j + v_1^{j-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & C[\frac{dv_1^j}{dt} + \frac{R-b}{R+b}\frac{dv_1^j}{dt}(t-\frac{2}{a})] \\ &= -\frac{1}{b}v_1^j + \frac{1}{b}\frac{R-b}{R+b}v_1^j(t-\frac{2}{a}) - g(v_1^j) - \frac{R-b}{R+b}g(v_1^j(t-\frac{2}{a})) \\ &+ \frac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}) + \frac{1}{R_o}\frac{R-b}{R+b}(v_1^{j+1} - 2v_1^j + v_1^{j-1}). \end{aligned} \tag{7.32}$$

In terms of matrices, we rewrite (7.32) as

$$\begin{aligned}&\frac{d}{dt}\left[x(t) - \alpha x(t-r)\right]\\ &= -\frac{1}{bC}Px(t) - \frac{1}{bC}\alpha Px(t-r) - \frac{1}{C}G(x(t)) + \frac{1}{C}\alpha G(x(t-r)),\end{aligned} \tag{7.33}$$

where

$$r = \frac{2}{a}, \quad x(t) = \begin{bmatrix} v_1^1(t) \\ \vdots \\ v_1^N(t) \end{bmatrix}, \quad G(x(t)) = \begin{bmatrix} g(v_1^1(t)) \\ \vdots \\ g(v_1^N(t)) \end{bmatrix},$$

$$\alpha = -\frac{R-b}{R+b}, \quad P = \begin{bmatrix} 1 + \frac{2b}{R_o} & -\frac{b}{R_o} & 0 & \cdots & 0 & -\frac{b}{R_o} \\ -\frac{b}{R_o} & 1 + \frac{2b}{R_o} & -\frac{b}{R_o} & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ -\frac{b}{R_o} & 0 & 0 & \cdots & -\frac{b}{R_o} & 1 + \frac{2b}{R_o} \end{bmatrix}.$$

Notice that the system (7.33) is a $\Gamma := D_N$-symmetric system in the following sense: consider Γ acting on $V := \mathbb{R}^N$ by permuting the coordinates of vectors $x = \begin{bmatrix} v^1 \\ \vdots \\ v^N \end{bmatrix} \in V$, then the system (7.33) is symmetric with respect to the Γ-action on V.

7.3 Hopf Bifurcation Results for Symmetric Configurations of Transmission Line Models

Motivated by the two generic models of symmetric couplings (cf. (7.29), (7.33)), we present a general symmetric system of functional differential equations and provide details in obtaining several important elements in computations of the associated bifurcation invariant, which are the prerequisite for the usage of our Maple© package.

7.3.1 Statement of the Problem

We are interested in studying the Hopf bifurcation problem in the following $\mathbb{R}$-parametrized system of symmetric functional differential equations

$$\begin{aligned}&\frac{d}{dt}\left[x(t)-\alpha Qx(t-r)\right]\\&=-P_1x(t)-\alpha QP_2x(t-r)-aG(x(t))+a\alpha QG(x(t-r)),\end{aligned} \tag{7.34}$$

where a and r are positive constants, α is the bifurcation parameter and $x(t)=[x^1(t),\dots,x^n(t)]^T\in\mathbb{R}^n$, $G(x(t))=[g(x^1(t)),\dots,g(x^n(t))]^T\in\mathbb{R}^n$. In addition, we assume

(H1) $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $g(0)=g'(0)=0$.
(H2) $V:=\mathbb{R}^n$ is a Γ-orthogonal representation, where Γ acts by permuting the coordinates of vectors $x\in V$.
(H3) (i) Q, P_1, P_2 are $n\times n$-matrices, which commute pairwisely.
 (ii) Q, P_1, P_2 commute with the Γ-action on V.
(H4) $|\alpha|\cdot\|Q\|<1$.

Remark 7.3.1. By (H3), Q, P_1, P_2 are pairwisely commuting matrices, thus they can be diagonalized simultaneously. In other words, Q, P_1, P_2 share the same eigenspaces with respect to a certain choice of a basis of V. We will use the symbols ξ, ζ and η to denote the eigenvalues of Q, P_1, and P_2 (respectively) corresponding to the same eigenvector $v\in V$. Further, assume that ζ and η satisfy the following

(H5) In the case $\zeta\eta>0$, $\sqrt{\zeta\eta}\neq\frac{2k+1}{2r}\pi$ for any $k\in\mathbb{Z}$.

By (H4), the system (7.34) satisfies (A5). It is clear that the system (7.34) is symmetric with respect to the Γ-action on V and $(\alpha,0)$ is a stationary point for all α. In this way, we are dealing with a Γ-symmetric system of neutral functional differential equations.

7.3.2 Isolated Centers

By linearizing the system (7.34) at $x=0$, we obtain

$$\frac{d}{dt}\left[x(t)-\alpha Qx(t-r)\right]=-P_1x(t)-\alpha QP_2x(t-r).$$

Substituting $x=e^{\lambda t}v$ for $\lambda\in\mathbb{C}$, $0\neq v\in V$, we have

$$\lambda e^{\lambda t}v-\alpha Q\lambda e^{\lambda(t-r)}v=-P_1e^{\lambda t}v-\alpha QP_2e^{\lambda(t-r)}v,$$

i.e.

$$\left[\lambda \mathrm{Id} - \alpha Q \lambda e^{-\lambda r} + P_1 + \alpha Q P_2 e^{-\lambda r}\right] v = 0.$$

Therefore, we have the following characteristic equation for the system (7.34)

$$\det\nolimits_{\mathbb{C}} \triangle_{(\alpha,0)}(\lambda) = 0, \tag{7.35}$$

where

$$\triangle_{(\alpha,0)}(\lambda) := (\lambda \mathrm{Id} - \alpha Q \lambda e^{-\lambda r}) + P_1 + \alpha Q P_2 e^{-\lambda r}.$$

To find isolated centers with their corresponding purely imaginary roots $i\beta$ for $\beta > 0$, we write (7.35) into algebraic equations using the eigenvalues of Q, P_1 and P_2. By Remark 7.3.1, when restricted to the same eigenspace of Q, P_1 and P_2, the characteristic equation (7.35) reduces to the following algebraic equation

$$(\lambda + \zeta)e^{\lambda r} - \alpha\xi(\lambda - \eta) = 0. \tag{7.36}$$

By replacing in (7.36) λ with $i\beta$ for some $\beta \neq 0$, and separating the real and imaginary parts, we obtain

$$\begin{cases} \zeta\cos(\beta r) - \beta\sin(\beta r) = -\eta\alpha\xi, \\ \zeta\sin(\beta r) + \beta\cos(\beta r) = \beta\alpha\xi, \end{cases} \tag{7.37}$$

which leads to

$$\tan(\beta r) = \begin{cases} \frac{\beta(\zeta+\eta)}{\beta^2 - \zeta\eta}, & \text{if } \beta^2 \neq \zeta\eta, \\ \infty, & \text{if } \beta^2 = \zeta\eta. \end{cases} \tag{7.38}$$

However, it can be verified that by (H5), the second case in (7.38) can not occur.

Hence, we have the following

$$\sin(\beta r) = \delta\frac{\beta(\zeta+\eta)}{\zeta^2+\beta^2}\sqrt{\frac{\zeta^2+\beta^2}{\eta^2+\beta^2}}, \quad \cos(\beta r) = \delta\frac{\beta^2-\zeta\eta}{\zeta^2+\beta^2}\sqrt{\frac{\zeta^2+\beta^2}{\eta^2+\beta^2}}. \tag{7.39}$$

where $\delta \in \{\pm 1\}$ depending on the range of βr. Also, observe that in the case $\xi = 0$, (7.37) does not permit any non-zero solution of β. So we suppose $\xi \neq 0$, then (7.37) yields:

$$\alpha = \frac{\delta}{\xi}\sqrt{\frac{\zeta^2+\beta^2}{\eta^2+\beta^2}}. \tag{7.40}$$

Using (7.40), we simplify (7.39) to

$$\sin(\beta r) = \frac{\alpha\xi\beta(\zeta + \eta)}{\zeta^2 + \beta^2}, \quad \cos(\beta r) = \frac{\alpha\xi(\beta^2 - \zeta\eta)}{\zeta^2 + \beta^2}. \tag{7.41}$$

Clearly, the assumption (A6) is satisfied for the system (7.34). We summarize the corresponding information in the statement following below (the needed arguments can be easily deducted from graphing (7.38)).

Lemma 7.3.2. *Given system (7.34) satisfying (H1) and (H3), fix a triple of reals ζ, η and ξ as in Remark 7.3.1 satisfying (H5). Then the equation*

$$\tan(\beta r) = \frac{\beta(\zeta + \eta)}{\beta^2 - \zeta\eta}$$

has infinitely many positive solutions β_k's ($k \in \mathbb{N}$), such that

(a) $0 < \beta_k < \beta_l$ *for* $k < l$;
(b) $\lim_{k \to \infty} \beta_k = \infty$;
(c) for each β_k, the point $(\alpha_k, 0)$ is an isolated center for system (7.34), where

$$\alpha_k = \frac{\delta}{\xi}\sqrt{\frac{\zeta^2 + \beta_k^2}{\eta^2 + \beta_k^2}}, \quad \delta = \pm 1.$$

Moreover,

(1) In the case $\zeta\eta > 0$, we put $k_o := \lfloor \frac{r\sqrt{\zeta\eta}}{\pi} + \frac{1}{2} \rfloor$, where the symbol $\lfloor \cdot \rfloor$ stands for the greatest integer function, we have
(1d) If $k_o = \frac{r\sqrt{\zeta\eta}}{\pi}$, then

$$\beta_k \in \begin{cases} (\frac{2k-1}{2r}\pi, \frac{k}{r}\pi) & \text{for } k < k_o \\ (\frac{k}{r}\pi, \frac{2k+1}{2r}\pi) & \text{for } k \geq k_o \end{cases}$$

(1d') Otherwise,

$$\beta_k \in \begin{cases} (\frac{2k-1}{2r}\pi, \frac{k}{r}\pi) & \text{for } k \leq k_o \\ (\frac{k-1}{r}\pi, \frac{2k-1}{2r}\pi) & \text{for } k > k_o \end{cases}$$

(2), In the case $\zeta\eta < 0$ and $\zeta + \eta < 0$, we have
(2d) $\beta_k \in (\frac{2k-1}{2r}\pi, \frac{k}{r}\pi)$ for $k \in \mathbb{N}$.
(3) In the case $\zeta\eta < 0$ and $\zeta + \eta > 0$, we have
(3d) If $\zeta + \eta \leq -\zeta\eta$, then $\beta_k \in (\frac{k}{r}\pi, \frac{2k+1}{2r}\pi)$ for $k \in \mathbb{N}$;
(3d') Otherwise, $\beta_k \in (\frac{k-1}{r}\pi, \frac{2k-1}{2r}\pi)$ for $k \in \mathbb{N}$.

7.3.3 Negative Spectrum

To use the computational formula (7.9), we need the information on the negative spectrum $\sigma_-(\overline{A})$ of the linear operator $\overline{A} = -\frac{1}{\beta_o} D_x f(\alpha_o, 0)$.

By (H1), we have that

$$-\frac{1}{\beta_o} D_x f(\alpha_o, 0) = \frac{1}{\beta_o}(P_1 + \alpha_o Q P_2) : \quad V \to V,$$

for each isolated center $(\alpha_o, 0)$.

To verify (A4), we will assume for a fixed triple of ξ, ζ, η that (cf. Remark 7.3.1)

(H6) $\zeta + \alpha_o \xi\eta \neq 0$.

The negative spectrum $\sigma_-(\overline{A})$ can be determined by

$$\begin{aligned}
\sigma_-(\overline{A}) &= \{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) \ : \ \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) < 0\} \\
&\overset{(7.40)}{=} \{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) \ : \ \frac{1}{\beta_o}(\zeta + \delta\sqrt{\frac{\zeta^2 + \beta_o^2}{\eta^2 + \beta_a^2}}\eta) < 0\} \\
&= \{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) \ : \ \zeta\sqrt{\eta^2 + \beta_o^2} + \delta\eta\sqrt{\zeta^2 + \beta_o^2} < 0\} \\
&= \{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) \ : \ \zeta + \delta\eta < 0\}. \qquad (7.42)
\end{aligned}$$

7.3.4 Isotypical Crossing Numbers

To proceed with the computational formula (7.9), we need to obtain the isotypical crossing numbers $\mathfrak{t}_{j,l}(\alpha_o, \beta_o)$, which can be computed by (cf. [15])

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o) = -\text{sign}\,(\frac{d}{d\alpha}u(\alpha_o))m_j(il\beta_o). \qquad (7.43)$$

To determine 'sign $(\frac{d}{d\alpha}u(\alpha_o))$', we substitute $\lambda = u + iv$ in (7.36) and separating the real and imaginary parts. Thus, we obtain

$$\begin{cases} e^{ur}(u + \zeta)\cos(vr) - e^{ur}v\sin(vr) = \xi\alpha(u - \eta), \\ e^{ur}(u + \zeta)\sin(vr) + e^{ur}v\cos(vr) = \xi\alpha v. \end{cases} \qquad (7.44)$$

By implicit differentiation of (7.44) with respect to α at α_o, $u = 0$, $v = \beta_o$, we obtain

$$\begin{cases} A\frac{du}{d\alpha}(\alpha_o) - B\frac{dv}{d\alpha}(\alpha_o) = -\eta\xi, \\ B\frac{du}{d\alpha}(\alpha_o) + A\frac{dv}{d\alpha}(\alpha_o) = \beta_o\xi, \end{cases} \tag{7.45}$$

where

$$\begin{cases} A = r(\zeta\cos(\beta_o r) - \beta_o\sin(\beta_o r)) + (\cos(\beta_o r) - \alpha_o\xi), \\ B = r(\zeta\sin(\beta_o r) + \beta_o\cos(\beta_o r)) + \sin(\beta_o r). \end{cases} \tag{7.46}$$

Substituting (7.41) into (7.46) leads to

$$\begin{cases} A = -\frac{\alpha_o\xi}{\zeta^2+\beta_o^2}\left[\eta r(\zeta^2+\beta_o^2) + \zeta(\zeta+\eta)\right], \\ B = \frac{\alpha_o\xi}{\zeta^2+\beta_o^2}\left[\beta_o r(\zeta^2+\beta_o^2) + \beta_o(\zeta+\eta)\right]. \end{cases} \tag{7.47}$$

Thus, it follows from (7.47) and (7.45) that

$$\begin{aligned} \frac{d\text{[illegible]}}{d\alpha}\text{[illegible]} &= \frac{1}{A^2+B^2}(-\eta\xi A + \beta_o\xi B) \\ &= \frac{\alpha_o\xi^2}{A^2+B^2}\left[r(\eta^2+\beta_o^2) + \frac{1}{\zeta^2+\beta_o^2}(\eta\zeta+\beta_o^2)(\zeta+\eta)\right]. \end{aligned} \tag{7.48}$$

Lemma 7.3.3. *Let $(\alpha_o, 0)$ be an isolated center for system (7.34) and $i\beta$ the corresponding characteristic root. Assume that for α close to α_o, the characteristic roots have the form $u(\alpha) + iv(\alpha)$. Assume, finally,*

(i) $r \geq 1$;
(ii) $\beta > 1$.

Then, we have

$$\operatorname{sign}\left(\frac{du}{d\alpha}(\alpha_o)\right) = \operatorname{sign}(\alpha_o).$$

Proof: Directly from (7.48), it suffices to show

$$\Upsilon(\eta,\zeta) := r(\eta^2+\beta_o^2) + \frac{1}{\zeta^2+\beta_o^2}(\eta\zeta+\beta_o^2)(\zeta+\eta) > 0.$$

Put

$$\Phi(\eta,\zeta) := \eta^2+\beta_o^2 + \frac{1}{\zeta^2+\beta_o^2}(\eta\zeta+\beta_o^2)(\zeta+\eta).$$

By assumption (i), $\Upsilon(\eta,\zeta) \geq \Phi(\eta,\zeta)$ for all η, ζ, thus we only need to show

$$\Phi(\eta,\zeta) > 0.$$

Case 1. If $\eta = 0$, $\Phi(0,\zeta) = \beta_o^2 + \frac{1}{\zeta^2+\beta_o^2}\beta_o^2\zeta = \beta_o^2\frac{\zeta^2+\zeta+\beta_o^2}{\zeta^2+\beta_o^2} \overset{(ii)}{>} 0.$

Case 2. If $\eta \neq 0$, then we can write $(\eta,\zeta) = (\eta, t\eta)$ for a unique $t \in \mathbb{R}$. Thus,

$$\Phi(\eta,t\eta) = \eta^2 + \beta_o^2 + \frac{1}{t^2\eta^2+\beta_o^2}(t\eta^2+\beta_o^2)(t+1)\eta.$$

Seeking a contradiction, assume

$$\Phi(\eta_o, t_o\eta_o) \leq 0 \tag{7.49}$$

at some $(\eta_o, t_o\eta_o)$ and put

$$\Psi(t) := \Phi(\eta_o, t\eta_o).$$

Since $\lim_{t\to\pm\infty}\Psi(t) = \eta_o^2 + \beta_o^2 + \eta_o \overset{(ii)}{>} 0$, it follows from (7.49) that $\Psi(t)$ has a non-positive minimum value at some t_{min}. An elementary calculus argument implies:

$$t_{min} = \begin{cases} \frac{\beta_o}{\eta_o} & \text{if} \quad \eta_o < 0, \\ -\frac{\beta_o}{\eta_o} & \text{if} \quad \eta_o > 0. \end{cases}$$

Thus,

$$\Psi(t_{min}) = \begin{cases} \eta_o^2 + \beta_o^2 + \frac{(\eta_o+\beta_o)^2}{2\beta_o} & \text{if} \quad \eta_o < 0, \\ \eta_o^2 + \beta_o^2 - \frac{(\eta_o-\beta_o)^2}{2\beta_o} & \text{if} \quad \eta_o > 0. \end{cases}$$

Clearly, in the case $\eta_o < 0$, $\Psi(t_{min}) > 0$, and for $\eta_o > 0$

$$\Psi(t_{min}) = \eta_o^2 + \beta_o^2 - \frac{(\eta_o-\beta_o)^2}{2\beta_o} \overset{(ii)}{>} \eta_o^2 + \beta_o^2 - \frac{(\eta_o-\beta_o)^2}{2} = \frac{(\eta_o+\beta_o)^2}{2} \geq 0,$$

and a contradiction arises, which asserts the conclusion. □

Thus, by Lemma 7.3.3 and (7.43), we have that

$$\mathfrak{t}_{j,l}(\alpha_o,\beta_o) = -\operatorname{sign}(\alpha_o)m_j(il\beta_o).$$

Without loss of generality, we can assume $\alpha_o < 0$. Therefore, we obtain

$$\mathfrak{t}_{j,l}(\alpha_o,\beta_o) = m_j(il\beta_o). \tag{7.50}$$

7.3.5 Computational Results

Similarly to Subsection 6.3.3—6.3.4, we keep the specific computational restraints, including only computing the first coefficient part $\omega(\lambda_o)_1$ of the local invariant and the condition (R1)—(R2).

Following the computational scheme outlined in Subsection 6.3.3, we prepare the input data sequence (cf. Subsection 6.3.4)

$$\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}, \quad \{\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r\}.$$

Then, using the computational formula

$$\omega(\alpha_0, \beta_o)_1 = \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i} \cdot m_j(i\beta_o) \deg_{\mathcal{V}_{j,1}}, \tag{7.51}$$

is equivalent to calling the command

$$\omega(\alpha_o, \beta_o)_1 = \texttt{showdegree}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r, \mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r).$$

In Appendix A4.2, we present quantative results for $\Gamma = D_4, A_5$.

8

Symmetric Hopf Bifurcation in Functional Partial Differential Equations

As the primary equivariant degree method proves to be effective in studying Hopf bifurcation problems in symmetric systems of ODEs, FDEs and NFDEs (cf. Chapters 6–7), in this chapter, we adapt this method to a setting appropriate for studying ***parabolic partial differential equations*** with delays.

Anticipating more potential applications, in Section 8.1, we establish a procedure for studying symmetric bifurcation in abstract parameterized coincidence equations involving unbounded Fredholm operators (depending continuously on a parameter). For technical reasons, it is convenient to consider such continuously parameterized family of Fredholm operators as a locally trivial Banach vector bundle over the parameter space. Using the vector bundle structure one can construct the so-called equivariant *resolvent*, which allows a conversion (in a standard way) of the coincidence problem into a fixed-point problem.

In Section 8.2, the standard abstract setting is adapted to a symmetric Hopf bifurcation problem in a system of functional parabolic partial differential equations. Section 8.3 is dealing with an application of the equivariant degree method to study the occurrence of symmetric Hopf bifurcation in the system of G.E. Hutchinson's parabolic equations with delay, modeling an interactive community ecosystem in a heterogeneous environment. A detailed analysis of equivariant spectral properties of the linearized system is presented, along with the important elements for the computational scheme. Using the Maple$^{\copyright}$ routines, we establish quantative results in a format of the associated local bifurcation invariants, providing the lower estimate of bifurcating branches of solutions and their symmetries, for $\Gamma = D_3, A_4$, which are listed in Appendix A4.3.

8.1 Bifurcation in a Parametrized Equivariant Coincidence Problem

Throughout this section, $G = \Gamma \times S^1$ with Γ being a compact Lie group.

8.1.1 Functional Setting

Let $\mathbb{E}$ and $\mathbb{F}$ be isometric Banach G-representations. Consider the space $\mathbb{E} \oplus \mathbb{F}$ equipped with the norm $\|(x,y)\|_{\mathbb{E}\oplus\mathbb{F}} := \|x\|_{\mathbb{E}} + \|y\|_{\mathbb{F}}$, where $\|\cdot\|_{\mathbb{E}}$ (resp. $\|\cdot\|_{\mathbb{F}}$) denotes the norm on $\mathbb{E}$ (resp. $\mathbb{F}$), together with the diagonal G-action on $\mathbb{E}\oplus\mathbb{F}$ by $g(x,y) := (gx, gy)$ for $g \in G$. Then, $\mathbb{E} \oplus \mathbb{F}$ becomes an isometric Banach G-representation.

For a (linear) operator L from $\mathbb{E}$ to $\mathbb{F}$, denote by $\mathrm{Dom}\,(L)$ and $\mathrm{Im}\,(L)$ the domain and the range of L respectively. An operator $L : \mathrm{Dom}\,(L) \subset \mathbb{E} \to \mathbb{F}$ is called *closed*, if its graph $\mathrm{Gr}\,(L) := \{(x, Lx) \;:\; x \in \mathrm{Dom}\,(L)\}$ is a closed subspace of $\mathbb{E}\oplus\mathbb{F}$. If L is additionally a G-equivariant (closed) operator, then the graph $\mathrm{Gr}\,(L)$ is G-invariant (closed) subspace of $\mathbb{E}\oplus\mathbb{F}$, which naturally becomes an isometric Banach G-subrepresentation of $\mathbb{E}\oplus\mathbb{F}$.

Denote by $\mathrm{Op}^G(\mathbb{E};\mathbb{F})$ the set of all closed G-equivariant operators from $\mathbb{E}$ to $\mathbb{F}$. Define a metric $\mathrm{dist}\,(\cdot,\cdot)$ on $\mathrm{Op}^G(\mathbb{E};\mathbb{F})$ by

$$\mathrm{dist}\,(L_1, L_2) := d_H\Big(S(Gr(L_1)), S(Gr(L_2))\Big),$$

where $L_i \in \mathrm{Op}^G(\mathbb{E};\mathbb{F})$, $S(Gr(L_i))$ denotes the unit sphere in $Gr(L_i)$ $(i = 1,2)$ and $d_H(\cdot,\cdot)$ is the *Hausdorff metric* on the space of all closed bounded subsets of $\mathbb{E}\oplus\mathbb{F}$. More precisely, for two closed bounded subsets X, Y of $\mathbb{E}\oplus\mathbb{F}$, define

$$D(X,Y) := \inf\{r > 0 \;:\; Y \subset X + B_r(\mathbb{E}\oplus\mathbb{F})\}.$$

Then, the *Hausdorff metric* d_H is given by

$$d_H(X,Y) := \max\{D(X,Y), D(Y,X)\}.$$

Recall the definition of the Fredholm operator as follows.

Definition 8.1.1. An operator $L : \mathrm{Dom}(L) \to \mathbb{F}$ defined on a dense subspace $\mathrm{Dom}\,(L) \subset \mathbb{E}$, is called a *Fredholm operator*, if

(i) L is a closed operator;

(ii) $\operatorname{Im}(L)$ is a closed subspace of $\mathbb{F}$;
(iii) $\dim \ker L < \infty$ and $\operatorname{codim} \operatorname{Im} L := \dim \mathbb{F}/\operatorname{Im} L < \infty$.

The number $\operatorname{ind}(L) := \dim \ker L - \operatorname{codim} L$ is called the *index* of L.

Let $\operatorname{Fr}_0^G(\mathbb{E};\mathbb{F}) \subset \operatorname{Op}^G(\mathbb{E};\mathbb{F})$ be the set of all G-equivariant *Fredholm operators* of index zero from $\mathbb{E}$ to $\mathbb{F}$. It can be verified that the set $\operatorname{Fr}_0^G(\mathbb{E};\mathbb{F})$ of all G-equivariant Fredholm operators of index zero is an open subset of $\operatorname{Op}^G(\mathbb{E};\mathbb{F})$ with respect to the metric $\operatorname{dist}(\cdot,\cdot)$ (cf. [15]). In particular, for any $L_{\lambda_o} \in \operatorname{Fr}_0^G(\mathbb{E};\mathbb{F})$ and sufficiently small $\varepsilon > 0$, we have that $\operatorname{dist}(L_\lambda, L_{\lambda_o}) < \varepsilon$ implies $L_\lambda \in \operatorname{Fr}_0^G(\mathbb{E};\mathbb{F})$. Moreover, if $\operatorname{dist}(L_\lambda, L_{\lambda_o})$ is sufficiently small, then there exists a G-equivariant linear isomorphism between $\operatorname{Gr}(L_\lambda)$ and $\operatorname{Gr}(L_{\lambda_o})$ (cf. [15]).

Consider a *continuous* family of G-equivariant Fredholm operators of index zero, $\{L_\lambda\}_{\lambda\in\mathcal{P}} \subset \operatorname{Fr}_0^G(\mathbb{E};\mathbb{F})$ parameterized by a topological space $\mathcal{P}$. Define a triple $(E, p_1, \mathcal{P})$ as follows. Put

$$E := \{(\lambda, x, y) \in \mathcal{P} \times (\mathbb{E} \oplus \mathbb{F}) \ : \ (x,y) \in \operatorname{Gr}(L_\lambda)\},$$

which is a G-invariant subset in $\mathcal{P} \times (\mathbb{E} \oplus \mathbb{F})$ (with the trivial G-action on $\mathcal{P}$). Define $p_1 : E \to \mathcal{P}$ by $p_1(\lambda, x, y) := \lambda$ for $(\lambda, x, y) \in E$, which is G-equivariant projection map onto $\mathcal{P}$. Notice that each $p_1^{-1}(\lambda) \simeq \operatorname{Gr}(L_\lambda)$ has a structure of an isometric Banach G-representation, for $\lambda \in \mathcal{P}$. Moreover, the continuity of ~~the fam~~ily $\{L_\lambda\}_{\lambda\in\mathcal{P}}$ implies that for any $\lambda_o \in \mathcal{P}$, there exists an open neighborhood U_o of λ_o such that for all $\lambda \in U_o$, $\operatorname{dist}(L_\lambda, L_{\lambda_o})$ is sufficiently small, which, in turn, gives rise to a G-equivariant linear isomorphism between $\operatorname{Gr}(L_\lambda)$ and $\operatorname{Gr}(L_{\lambda'})$. Indeed, it was shown in [54] that $(E, p_1, \mathcal{P})$ is a locally trivial G-vector bundle.

Further, it turns out to be convenient to identify $(E, p_1, \mathcal{P})$ with yet another G-vector bundle defined as follows. For $L \in \operatorname{Fr}^G(\mathbb{E};\mathbb{F})$, define the *graph norm* on $\operatorname{Dom}(L)$ by

$$\|x\|_L := \|x\|_{\mathbb{E}} + \|Lx\|_{\mathbb{F}}, \quad x \in \operatorname{Dom}(L).$$

Consequently, $(\operatorname{Dom}(L), \|\cdot\|_L)$ is canonically G-isomorphic to $(\operatorname{Gr}(L), \|\cdot\|_{\mathbb{E}\oplus\mathbb{F}})$. For convenience, we write

$$\mathbb{E}_L := (\operatorname{Dom}(L), \|\cdot\|_L),$$

which is an isometric Banach G-representation, under the identification with $(\mathrm{Gr}(L), \|\cdot\|_{\mathbb{E}\oplus\mathbb{F}})$.

Define a triple $(\mathcal{E}, p, \mathcal{P})$ by the following. Put

$$\mathcal{E} := \{(\lambda, x) \in \mathcal{P} \times \mathbb{E} \;:\; x \in \mathbb{E}_{L_\lambda}\},$$

and define $p : \mathcal{E} \to \mathcal{P}$ by $p(\lambda, x) := \lambda$. Notice that each $p^{-1}(\lambda) \simeq \mathbb{E}_{L_\lambda}$, for $\lambda \in \mathcal{P}$. Through the identification between $\mathbb{E}_{L_\lambda}$ and $\mathrm{Gr}(L_\lambda)$, one argues that $(\mathcal{E}, p, \mathcal{P})$ is indeed a locally trivial G-vector bundle. Moreover, the map $\psi : E \to \mathcal{E}$ given by $\psi(\lambda, x, y) := (\lambda, x)$ provides a G-vector bundle isomorphism between $(E, p_1, \mathcal{P})$ and $(\mathcal{E}, p, \mathcal{P})$.

We are now in a position to formulate a *parameterized G-equivariant coincidence problem* (cf. [113]). Define a G-vector bundle morphism $L : \mathcal{E} \to \mathbb{F}$ by

$$L(\lambda, u) = L_\lambda u, \quad (\lambda, u) \in \mathcal{E}, \tag{8.1}$$

where $\mathbb{F}$ is viewed as a trivial G-vector bundle over a singleton. Given a completely continuous G-equivariant map $F : \mathcal{E} \to \mathbb{F}$, we are interested in finding solutions to the following parameterized G-equivariant coincidence problem

$$L_\lambda u = F(\lambda, u), \quad (\lambda, u) \in \mathcal{E}|_{X \times \mathrm{Dom}(L_\lambda)}, \tag{8.2}$$

where $X \subset \mathcal{P}$ is an appropriately chosen subset on which it is possible to convert (8.2) to a G-equivariant fixed-point problem.

The following notion of an equivariant resolvent is a key to convert (8.2) to a G-equivariant fixed-point problem.

Definition 8.1.2. Let $X \subset \mathcal{P}$ be a subset and L be given by (8.1). An *equivariant resolvent* of L over X is a G-vector bundle morphism $K : \mathcal{E}|_{X \times \mathbb{E}} \to \mathbb{F}$ such that

(i) for every $\lambda \in X$, $K_\lambda : \mathbb{E}_{L_\lambda} \to \mathbb{F}$ is a finite-dimensional operator;
(ii) for every $\lambda \in X$, $L_\lambda + K_\lambda : \mathbb{E}_{L_\lambda} \to \mathbb{F}$ is a linear G-isomorphism.

Denote by $\mathcal{R}^G(L, X)$ the set of all equivariant resolvents of L over X. In contrast to the non-equivariant case, it might happen that $\mathcal{R}^G(L, \{\lambda_o\}) = \emptyset$, for some $\lambda_o \in \mathcal{P}$. In general, even $\mathcal{R}^G(L, \{\lambda\}) \neq \emptyset$ for each $\lambda \in X$, it is possible that $\mathcal{R}^G(L, X) = \emptyset$. However, we have the following

Lemma 8.1.3. *(cf. [113]) Let $X \subset \mathcal{P}$ be a compact contractible set containing a point λ^* such that $\mathcal{R}^G(L, \{\lambda^*\}) \neq \emptyset$. Then, $\mathcal{R}^G(L, X) \neq \emptyset$.*

Throughout this section, we assume that

(H1) There exists a compact subset $X \subset \mathcal{P}$ such that $\mathcal{R}^G(L, X) \neq \emptyset$.

Fix an equivariant resolvent $K \in \mathcal{R}^G(L, X)$. For each $\lambda \in X$, put

$$R_\lambda := (L_\lambda + K_\lambda)^{-1}, \tag{8.3}$$

which is a linear G-isomorphism. Therefore, (8.2) can be converted to a G-equivariant fixed-point problem

$$y = \mathcal{F}(\lambda, y), \quad (\lambda, y) \in X \times \mathbb{F}, \tag{8.4}$$

where

$$\mathcal{F}(\lambda, y) = F(\lambda, R_\lambda y) + K_\lambda(R_\lambda y), \quad (\lambda, y) \in X \times \mathbb{F}.$$

By the compactness of X (cf. (H1)), $\mathcal{F} : X \times \mathbb{F} \to \mathbb{F}$ is a completely continuous map.

8.1.2 Bifurcation Invariant for the Equivariant Coincidence Problem

Let $\mathcal{P} = \mathbb{R} \times \mathbb{R}_+$ and $\mathbb{E}, \mathbb{F}$ isometric Banach G-representations. Suppose that $\{L_\lambda\}_{\lambda \in \mathcal{P}}$ is a continuous family of G-equivariant Fredholm operators of index zero satisfying (H1). Fix $K \in \mathcal{R}^G(L, X)$ and let R_λ be defined by (8.3), for $\lambda \in X$.

Motivated by the parametrized parabolic system to be discussed in the next section, we assume that

(H2) (i) there exists another real isometric Banach G-representation $\widehat{\mathbb{E}}$ and an injective G-vector bundle morphism $J : \mathcal{E} \to \mathcal{P} \times \widehat{\mathbb{E}}$ such that $J_\lambda := J(\lambda, \cdot)$ is a compact linear operator for every $\lambda \in \mathcal{P}$;
(ii) there exists an equivariant C^1-map $\widehat{F} : \mathcal{P} \times \widehat{E} \to \mathbb{F}$.

Define

$$F := \widehat{F} \circ J, \tag{8.5}$$

which is a G-equivariant completely continuous map by (H2)(i). Consider the coincidence problem (8.2) with F defined by (8.5). Assume, in addition, that there exists a two-dimensional submanifold $M \subset \mathcal{P} \times \mathbb{E}^G$ satisfying:

(H3) M is a subset of the solution set of (8.2);

(H4) for $(\lambda_o, u_o) \in M$, there exists an open neighborhood $U_{\lambda_o} \subset X$ of λ_o and $U_{u_o} \subset \mathbb{E}^G$ of u_o and a C^1-map $\chi : U_{\lambda_o} \to \mathbb{E}^G$ such that

$$M \cap (U_{\lambda_o} \times U_{u_o}) = Gr(\chi).$$

We call each $(\lambda, u) \in M$ a *trivial solution* of (8.2). All the other solutions will be called *nontrivial*. A point $(\lambda_o, u_o) \in M$ is called a *bifurcation point*, if in each neighborhood of (λ_o, u_o), there exists a nontrivial solution of (8.2). We are interested in studying the bifurcation problem of (8.2), including establishing the existence of nontrivial solutions bifurcating from the surface M.

Notice that (λ, u) is a solution of the system (8.2) if and only if (λ, y) is a solution of the system (8.4), for $y = (L_\lambda + K_\lambda)u$. Moreover, the set of the trivial solutions to (8.4) can be expressed by

$$\widetilde{M} := \{(\lambda, y) \in X \times \mathbb{F} : (\lambda, R_\lambda(y)) \in M\}.$$

Thus, the assumption (H4) is equivalent to

(H4)' if $(\lambda_o, y_o) \in \widetilde{M}$, then there exists an open neighborhood $U_{\lambda_o} \subset X$ of λ_o and $U_{y_o} \in \mathbb{F}^G$ of y_o and a C^1-map $\widetilde{\chi} : U_{\lambda_o} \to \mathbb{F}^G$ such that

$$\widetilde{M} \cap (U_{\lambda_o} \times U_{y_o}) = Gr(\widetilde{\chi}).$$

Define the projection map $\pi : X \times \mathbb{F} \to \mathbb{F}$ by $\pi(\lambda, y) = y$. Then, the system (8.4) can be reformulated as

$$(\pi - \mathcal{F})(\lambda, y) = 0, \quad (\lambda, y) \in X \times \mathbb{F}, \tag{8.6}$$

By the assumption (H2), $\pi - \mathcal{F}$ is a G-equivariant completely continuous field of class C^1. Consider the differential operator

$$D_y(\pi - \mathcal{F}) = \mathrm{Id} - \Big(D_u F(\lambda, R_\lambda(y)) R_\lambda + K_\lambda R_\lambda\Big);$$

which is a bounded G-equivariant Fredholm operator of index zero (cf. (H2)). Notice that, by implicit function theorem, if $(\lambda_o, y_o) \in \widetilde{M}$ is a bifurcation point, then $D_y(\pi - \mathcal{F})$ is not an isomorphism at (λ_o, y_o). A point $(\lambda_o, y_o) \in \widetilde{M}$ is called *L-singular*, if $D_y(\pi - \mathcal{F})$ is not an isomorphism at (λ_o, y_o). An L-singular point (λ_o, y_o) is *isolated*, if it is the only L-singular point in some neighborhood of (λ_o, y_o) in $\widetilde{M}$.

We assume that

(H5) there exists an isolated L-singular point $(\lambda_o, y_o) \in \widetilde{M}$.

Given an isolated L-singular point $(\lambda_o, y_o) \in \widetilde{M}$, following the same construction as in Subsection 6.1.5, we define an isolating neighborhood $U(r) \subset X \times \mathbb{F}$ around (λ_o, y_o) and a G-equivariant auxiliary function $\varsigma : \overline{U}(r) \to \mathbb{R}$. Based on the auxiliary function, a completely continuous field $\mathfrak{F}_\varsigma : \overline{U}(r) \to \mathbb{R} \oplus \mathbb{F}$ is constructed to define a local bifurcation invariant $\omega(\lambda_o, y_o)$ using the twisted primary equivariant degree.

More precisely, take a neighborhood $\mathcal{D}_{\lambda_o}$ of (λ_o, y_o) in $\widetilde{M}$ such that (λ_o, y_o) is the only L-singular point in $\mathcal{D}_{\lambda_o}$ and $\overline{\mathcal{D}}_{\lambda_o} \subset \widetilde{M} \cap (U_{\lambda_o} \times U_{y_o})$ (cf. (H4)').

For a small $r > 0$, define $U(r) \subset X \times \mathbb{F}$ by

$$U(r) := \{(\lambda, y) \in X \times \mathbb{F} \;:\; (\lambda, \widetilde{\chi}(\lambda)) \in \mathcal{D}_{\lambda_o},\ \|y - \widetilde{\chi}(\lambda)\| < r\}. \tag{8.7}$$

Put

$$\partial U_0 := \{(\lambda, y) \in \overline{U(r)} : (\lambda, \widetilde{\chi}(\lambda)) \in \partial \mathcal{D}_{\lambda_o}\} \subset \partial U(r).$$

By (H5) and the implicit function theorem, we can choose $r > 0$ sufficiently small that

$$y - \mathcal{F}(\lambda, y) \neq 0, \quad \text{for } (\lambda, y) \in \partial U_0 \setminus \widetilde{M}.$$

Let $\varsigma : \overline{U(r)} \to \mathbb{R}$ be a G-invariant auxiliary function such that

$$\begin{cases} \varsigma(\lambda, y) > 0, & \text{if } \|y - \widetilde{\chi}(\lambda)\| = r, \\ \varsigma(\lambda, y) < 0. & \text{if } (\lambda, y) \in \mathcal{D}_{\lambda_o}. \end{cases} \tag{8.8}$$

Define the map $\mathfrak{F}_\varsigma : \overline{U(r)} \to \mathbb{R} \oplus \mathbb{F}$ by

$$\mathfrak{F}_\varsigma(\lambda, y) := (\varsigma(\lambda, y), (\pi - \mathcal{F})(\lambda, y)), \tag{8.9}$$

which is clearly a $U(r)$-admissible G-equivariant completely continuous vector field.

Definition 8.1.4. Let $U(r), \varsigma, \mathfrak{F}_\varsigma$ be defined by (8.7), (8.8) and (8.9) respectively. We call

$$\omega(\lambda_o, y_o) := G\text{-Deg}^t(\mathfrak{F}_\varsigma, U(r)) \in A_1^t(G) \tag{8.10}$$

the *local bifurcation invariant* for the parametrized equivariant coincidence problem (8.2) at (λ_o, y_o).

The following theorem provides us with a sufficient condition for the existence of nontrivial solutions of (8.2) bifurcating from (λ_o, y_o). For the ideas of the proof, we refer to [15].

Theorem 8.1.5. (LOCAL BIFURCATION THEOREM) *Suppose that the assumptions (H1)—(H5) are satisfied, $\omega(\lambda_o, u_o)$ is given by (8.10) (with $\mathfrak{F}_\varsigma$ defined by (8.9), $U(r)$ by (8.7) and ς satisfying (8.8)). If*

$$\omega(\lambda_o, y_o) = \sum_{(H)} n_H\,(H) \neq 0,$$

i.e., there is $n_{H_o} \neq 0$ for some orbit type (H_o), then there exists a branch of non-trivial solutions (λ, y) to the equation (8.2) bifurcating from (λ_o, y_o) such that $G_y \supset H_o$.

8.2 Hopf Bifurcation in Symmetric Systems of Functional Parabolic Differential Equations

Let $V := \mathbb{R}^n$ be an orthogonal Γ-representation and $\Omega \subset \mathbb{R}^m$ an open bounded set such that $\partial\Omega$ is C^2-smooth. The space $L^2(\mathbb{R} \times \overline{\Omega}; V)$ of L^2-integrable V-valued functions is an isometric Banach Γ-representation with the Γ-action given by

$$(\gamma u)(t, x) = \gamma(u(t, x)), \quad u \in L^2(\mathbb{R} \times \overline{\Omega}; V),\ \gamma \in \Gamma.$$

8.2.1 Statement of the Problem

Consider a system of functional parabolic differential equations on $\mathbb{R} \times \overline{\Omega}$

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + P(\alpha, x)u = f(\alpha, u_t)(x) & (t, x) \in \mathbb{R} \times \Omega, \\ B(\alpha, x)u(t, x) = 0 & (t, x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{8.11}$$

where $u \in L^2(\mathbb{R} \times \overline{\Omega}; V)$ satisfies appropriate differentiability requirements,[*] $u_t(\theta, x) := u(t + \theta, x)$ for $\theta \in [-\tau, 0]$ ($\tau > 0$ is a fixed constant), $\alpha \in \mathbb{R}$ is a (bifurcation) parameter, $f : \mathbb{R} \times C([-\tau, 0]; L^2(\Omega; V)) \to L^2(\Omega; V)$ is a map of class C^1, which is bounded on bounded sets, $P(\alpha, x) = [P_i(\alpha, x)]_{i=1}^n$ is a vector with components being second-order uniformly elliptic operators, i.e.

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[*] u is weakly differentiable with respect to $t \in \mathbb{R}$ and has weak derivatives of order 2 with respect to $x \in \Omega$.

$$P_i(\alpha, x) = \nabla^T A_i(\alpha, x)\nabla + a_i(\alpha, x),$$

with $A_i(\alpha, x)$ being a continuously differentiable (with respect to α and x) $n \times n$ symmetric positive definite matrix satisfying

$$\exists c_1, c_2 > 0 \;\; \forall (\alpha, x) \in \mathbb{R} \times \overline{\Omega} \;\; \forall y \in V' \quad c_1\|y\| \le y^T A_i(\alpha, x) y \le c_2\|y\|,$$

where ∇ stands for the gradient operator, and $a_i(\alpha, x)$ is continuous. The boundary operator $B(\alpha, x)$ is defined by either (Dirichlet conditions)

$$B(\alpha, x)u(t, x) = u(t, x)$$

or (mixed Dirichlet/Neumann conditions)

$$B(\alpha, x)u(t, x) = b(\alpha, x)u(t, x) + \frac{\partial}{\partial n}(\alpha, x)\, u(t, x),$$

where $b \in C^1(\mathbb{R} \times \partial\Omega; \mathbb{R})$, $\frac{\partial}{\partial n}(\alpha, x) = [\nu^T(x)A_i(\alpha, x)\nabla]_{i=1}^n$ and $\nu(x)$ is the outward normal vector to $\partial\Omega$ at x.

We assume that

(C1) the operators P, B and the map f are Γ-equivariant.

Use the standard identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and introduce the following notation

$$\mathcal{H}^{1,2}_{B(\alpha)} = \{\varphi \in H^{1,2}(S^1 \times \Omega; V) \;:\; B(\alpha, x)\varphi = 0\}, \tag{8.12}$$

where $H^{k,l}(S^1 \times \Omega; V)$ stands for the Sobolev space of V-valued functions with weak L^2-integrable derivatives of order k in S^1 and of order l in Ω. Put

$$\begin{aligned} \mathbb{E} = \mathbb{F} &:= L^2(S^1 \times \Omega; V), \\ \mathcal{P} &:= \mathbb{R} \times \mathbb{R}_+, \\ \widehat{\mathbb{E}} &:= C\big(S^1; L^2(\Omega; V)\big), \end{aligned} \tag{8.13}$$

where $\widehat{\mathbb{E}}$ is equipped with the usual supremum norm.

8.2.2 Normalization of the Period

Let $\beta := \frac{2\pi}{p}$ and $v(t,x) := u(\frac{1}{\beta}\, t, x)$. Then, the problem (8.11) of finding a p-periodic solution is equivalent to finding a 2π-periodic solution (α, β, v) of the system

$$\begin{cases} \frac{\partial}{\partial t}\, v(t,x) + \dfrac{1}{\beta} P(\alpha, x)v = \dfrac{1}{\beta} f(\alpha, v_{t,\beta})(x) & (t,x) \in \mathbb{R} \times \Omega, \\ B(\alpha, x)v(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \\ v(t,x) = v(t + 2\pi, x) & (t,x) \in \mathbb{R} \times \Omega, \end{cases} \tag{8.14}$$

where

$$v_{t,\beta}(\theta, x) := v(t + \beta\theta, x) \quad \text{for} \quad (\theta, x) \in [-\tau, 0] \times \Omega.$$

8.2.3 Setting in Functional Spaces

Following the discussion in Section 8.1.1, we reformulate the system (8.14) as a parameterized equivariant coincidence problem.

For $\lambda := (\alpha, \beta) \in \mathcal{P}$, define the subspace

$$\mathrm{Dom}\big(L_\lambda\big) := \{u \in \mathbb{E} \;:\; u \in \mathcal{H}^{1,2}_{B(\alpha)}\},$$

and the operator $L_\lambda :\ \mathrm{Dom}(L_\lambda) \subset \mathbb{E} \to \mathbb{E}$ by

$$L_\lambda v(t,x) := \frac{\partial}{\partial t}\, v(t,x) + \frac{1}{\beta}\, P(\alpha, x)v,$$

(cf. (8.12), (8.13) and (8.14)).

Notice that $\mathbb{E}$, $H^{1,2}(S^1 \times \Omega; V)$ and $\widehat{\mathbb{E}}$ are isometric Banach G-representations, where S^1 acts in a standard way by shifting the time argument t. It is also clear (cf. [127]) that each (unbounded) linear operator L_λ, for $\lambda \in \mathcal{P}$, is a closed G-equivariant Fredholm operator of index zero. Moreover, the orthogonal projection on the (finite-dimensional) kernel of L_λ is a G-equivariant resolvent K of L_λ. Therefore, $R^G(L, \{\lambda\}) \neq \emptyset$ for any $\lambda \in \mathcal{P}$. Thus, by Lemma 8.1.3, the condition (H1) is satisfied for every compact subset $X \subset \mathcal{P}$.

On the other hand, since $\frac{1}{\beta} f(\alpha, v_{t,\beta}) \in L^2(\Omega; V)$ for $v_{t,\beta} \in C\big([-\tau, 0]; L^2(\Omega; V)\big)$, we have the continuous map $N_f : \mathcal{P} \times \widehat{\mathbb{E}} \to L^2(\Omega, V)$ with

$$N_f(\alpha,\beta,v)(t) := \frac{1}{\beta} f(\alpha, v_{t,\beta}).$$

Define $\widehat{F} : \mathcal{P} \times \widehat{\mathbb{E}} \to \mathbb{F}$ by

$$\widehat{F}(\lambda, v)(t,x) := j \circ N_f(\alpha,\beta,v)(t)(x) = \frac{1}{\beta} f(\alpha, v_{t,\beta})(x), \quad \lambda = (\alpha,\beta),$$

where j denotes the natural embedding $\widehat{\mathbb{E}} \hookrightarrow \mathbb{F}$. The continuous differentiability of f implies that $\widehat{F}$ is continuously differentiable. Since the following composition of the embeddings

$$H^{1,2}(S^1 \times \Omega; V) \hookrightarrow H^{\frac{2}{3},0}(S^1 \times \Omega; V) \hookrightarrow C\big(S^1; L^2(\Omega; V)\big) = \widehat{\mathbb{E}}$$

is compact (cf. [127]), we have the following embedding

$$J : \mathcal{E} \longrightarrow \mathcal{P} \times \widehat{\mathbb{E}}$$

where $J_\lambda : \mathbb{E}_{L_\lambda} \to \widehat{\mathbb{E}}$ is a compact operator for all $\lambda \in \mathcal{P}$. Thus $\widehat{F}$ and J satisfy the condition (H2) from Section 8.1.2. In particular, $F : \mathcal{E} \to \mathbb{F}$ defined by $F = \widehat{F} \circ J$ is a G-equivariant completely continuous map of class C^1.

As a consequence, we obtain that finding a periodic solution $v \in H^{1,2}(S^1 \times \Omega; V)$ for the system (8.14) is equivalent to solving the following parameterized coincidence problem (cf. (8.2))

$$L_\lambda v = F(\lambda, v), \quad \lambda \in X, \tag{8.15}$$

where X is a given compact subset of $\mathcal{P}$.

8.2.4 Γ-Symmetric Steady-State Solutions

Observe that the constant (with respect to t) functions $u(t,x) \in H^{1,2}(S^1 \times \Omega; V)$ can be identified with functions $u(x) \in H^2(\Omega; V)$, which is the space of V-valued functions with weak L^2-integrable derivatives of order 2 in Ω. Clearly, for $u(x) \in H^2(\Omega; V)$, we have $u_t(\theta, x) \equiv u(\theta, x)$ for $t \in \mathbb{R}$.

To describe the set of trivial solutions to (8.11), we introduce the following

Definition 8.2.1. A solution (α_o, u_o) of (8.11) is called a *Γ-symmetric steady-state* solution, if it satisfies

(i) $u_o \in H^2(\Omega, V)$;

(ii) $\gamma u_o = u_0$ for all $\gamma \in \Gamma$;

(iii) $\begin{cases} P(\alpha_o, x)u_o = f(\alpha_o, u_o)(x) & \text{in } \Omega, \\ B(\alpha_o, x)u_o = 0 & \text{on } \partial\Omega. \end{cases}$

Denote the following spaces by

$$\begin{aligned} \mathfrak{B}_{\alpha_o} &:= \{\omega \in H^2(\Omega; V) : B(\alpha_o, x)\omega = 0\}, \\ \mathfrak{B}^c_{\alpha_o} &:= \{\omega \in H^2(\Omega; V^c) : B(\alpha_o, x)\omega = 0\}, \\ \mathfrak{C}_\tau &:= C([-\tau, 0]; L^2(\Omega; V)), \\ \mathfrak{C}^c_\tau &:= C([-\tau, 0]; L^2(\Omega; V^c)). \end{aligned}$$

Notice that we can view $L^2(\Omega; V) \subset \mathfrak{C}_\tau$ is the subspace of constant $L^2_\tau(\Omega; V)$-valued functions. Similarly, $L^2(\Omega; V^c) \subset \mathfrak{C}^c_\tau$ is the subspace of constant $L^2(\Omega; V^c)$-valued functions.

Put $\bar{f} := f|_{\mathbb{R}\times L^2(\Omega;V)}$ and

$$\mathcal{L}_{\alpha_o} := P(\alpha_o, x) - D_u\bar{f}(\alpha_o, u_o) : \mathfrak{B}_{\alpha_o} \to L^2(\Omega; V). \tag{8.16}$$

We will use the same symbols to denote the complexified operators $P(\alpha, x)$, $D_u\bar{f}(\alpha_o, u_o)$ and $B(\alpha_o, x)$.

Definition 8.2.2. A Γ-symmetric steady-state solution (α_o, u_o) of (8.11) is called *nonsingular*, if $0 \notin \sigma(\mathcal{L}_{\alpha_o})$, where $\sigma(\mathcal{L}(\alpha_o))$ is the spectrum of $\mathcal{L}_{\alpha_o}$.

Assume that

(C2) there exists a nonsingular Γ-symmetric steady-state solution (α_o, u_o) of (8.11).

Thus, by implicit function theorem, there exists a small $\eta > 0$ and a C^1-function $u(\alpha)$ for $|\alpha - \alpha_o| < \eta$ such that $(\alpha, u(\alpha))$ is a Γ-symmetric steady-state solution to (8.11) for each α.

Throughout the rest of this section, we assume that

$$\{(\alpha, u(\alpha)) \; : \; |\alpha - \alpha_o| < \eta\} \subset \mathcal{P} \times \mathbb{E}^G,$$

is a fixed family of steady-state Γ-symmetric solutions through (α_o, u_o), and each $(\alpha, \beta, u(\alpha))$ is called a *trivial solution* of (8.11). Moreover, we can define the map $\chi : (\alpha_o - \eta, \alpha_o + \eta) \times \mathbb{R} \to \mathbb{E}^G$ by $\chi(\alpha, \beta) = (\alpha, \beta, u(\alpha))$. Consequently, the set of Γ-symmetric steady-state solutions to (8.11) gives rise to a manifold $M \subset \mathcal{P} \times \mathbb{E}^G$, which is defined locally by

$$M := \{(\alpha, \beta, u(\alpha)) : \alpha \in (\alpha_o - \eta, \alpha_o + \eta),\ \beta \in \mathbb{R}\}$$

and M satisfies (H3) and (H4).

8.2.5 Characteristic Equation

Let $(\alpha, u(\alpha))$ be a nonsingular Γ-symmetric steady-state solution of (8.11) near (α_o, u_o). The linearization of (8.11) at $(\alpha, u(\alpha))$ leads to the *characteristic equation*

$$\triangle_{\alpha;u(\alpha)}(\lambda)w := \lambda w + P(\alpha, x)w - D_u\bar{f}(\alpha, u(\alpha))(e^{\lambda\cdot}w) = 0, \quad \lambda \in \mathbb{C}, \qquad (8.17)$$

where the *characteristic operator* $\triangle_{\alpha;u(\alpha)}(\lambda) : \mathfrak{B}^c_\alpha \to L^2(\Omega; V^c)$ is defined using the complexifications of $P(\alpha, x)$ and $D_u\bar{f}(\alpha, u(\alpha))$.

Notice that $\triangle_{\alpha;u(\alpha)}(\lambda)$ is a closed (unbounded) Fredholm operator of index zero from $L^2(\Omega; V^c)$ to itself. Indeed, the embedding $\mathfrak{B}^c_\alpha \hookrightarrow L^2(\Omega; V^c)$ is compact with respect to the H^2-norm on $\mathfrak{B}^c_\alpha$. The operator $P(\alpha, x)$ being elliptic self-adjoint, is a (bounded) Fredholm operator of index zero, and $D_u\bar{f}(\alpha, u(\alpha))(e^{\lambda\cdot}\cdot)$ is a bounded linear operator. Therefore, $\triangle_{\alpha;u(\alpha)}(\lambda)$ is a (bounded) Fredholm operator of index zero from $\mathfrak{B}^c_\alpha$ (equipped with the H^2-norm) to $L^2(\Omega; V^c)$. Consequently, $\triangle_{\alpha;u(\alpha)}(\lambda)$ is a closed (unbounded) Fredholm operator of index zero from $L^2(\Omega; V^c)$ to itself.

Similar as in Subsection 6.1.1, we define the characteristic root, center and isolated center.

Definition 8.2.3. A number $\lambda \in \mathbb{C}$ is called a *characteristic root* of the system (8.11) at a Γ-symmetric steady-state solution $(\alpha, u(\alpha))$, if $\ker \triangle_{\alpha;u(\alpha)}(\lambda) \neq \{0\}$. A nonsingular Γ-symmetric steady-state solution (α_o, u_o) is a *center*, if it has a purely imaginary characteristic root $i\beta_o$ for $\beta_o > 0$. A center (α_o, u_o) is called *isolated*, if it is the only center in some neighborhood of (α_o, u_o) in $\mathbb{R} \oplus L^2(\Omega; V)$.

We assume that

(C3) there exists an isolated center $(\alpha_o, u_o) \in \mathbb{R} \oplus L^2(\Omega; V)$ such that $i\beta_o$ is a characteristic root of (8.11) for $\beta_o > 0$, i.e. $\ker \triangle_{\alpha_o;u_o}(i\beta_o) \neq \{0\}$.

By (C3), the condition (H5) from Subsection 8.1.2 is satisfied. Also, (C3) provides a necessary condition for the occurrence of the Hopf bifurcation at (α_o, u_o). The condition (C2) excludes the appearance of the "steady-state" bifurcation.

Denote by $\sigma_\alpha \subset \mathbb{R}$ the spectrum of $P(\alpha, x) : \mathfrak{B}^c_\alpha \to L^2(\Omega; V^c)$. Since $P(\alpha, x)$ is a uniformly elliptic differential operator, the spectrum σ_α is discrete and each eigenvalues $\mu^\alpha_k \in \sigma_\alpha$ is real and of finite multiplicity. Suppose that

$$\mu^\alpha_0 < \mu^\alpha_1 < \cdots < \mu^\alpha_k < \ldots .$$

For any fixed $r > 0$, observe that $ir \notin \sigma_\alpha$. Thus, we define an auxiliary operator $S : L^2(\Omega; V^c) \to L^2(\Omega; V^c)$ by

$$Sw = irw, \quad w \in L^2(\Omega; V^c),$$

which is a Γ-equivariant resolvent of $P(\alpha, x)$. In particular, inverse map

$$\widetilde{R}_{\alpha,r} := [P(\alpha, x) + S]^{-1}$$

is a bounded Γ-equivariant operator from $L^2(\Omega; V^c)$ to $\mathfrak{B}^c_\alpha$ (equipped with the H^2-norm). Moreover, since the embedding $\mathfrak{B}^c_\alpha \hookrightarrow L^2(\Omega; V^c)$ is compact, we obtain that $\widetilde{R}_{\alpha,r}$ is a compact Γ-equivariant operator from $L^2(\Omega; V^c)$ to itself.

Using the inverse operator $\widetilde{R}_{\alpha,r}$, (8.17) can be re-written as

$$\widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)w := w + (\lambda - ir)\widetilde{R}_{\alpha,r}(w) - D_u\bar{f}(\alpha, u(\alpha))\big(e^{\lambda\cdot}\widetilde{R}_{\alpha,r}(w)\big) = 0. \quad (8.18)$$

It is clear that $\lambda \in \mathbb{C}$ is a characteristic root of the system (8.11) at the steady-state solution $(\alpha, u(\alpha))$ if and only if $\ker \widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda) \neq \{0\}$. Since $\widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)$ is an analytic function in λ (cf. [180]), all the characteristic roots λ are isolated. Moreover, $\widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)$ is a Γ-equivariant compact field, thus it is a bounded Γ-equivariant Fredholm operator of index zero.

Denote by $E^\alpha_k \subset L^2(\Omega; V^c)$ the eigenspace of $P(\alpha, x)$ corresponding to $\mu^\alpha_k \in \sigma_\alpha$. Let $\mathfrak{p}^\alpha_k : L^2(\Omega; V^c) \to E^\alpha_k$ be the orthogonal projection map. Consequently, for every $w \in L^2(\Omega; V^c)$ we can write $w = \sum_{k=0}^{\infty} \mathfrak{p}^\alpha_k(w)$. Then, (8.18) is equivalent to

$$\sum_{k=0}^{\infty}\left[\mathfrak{p}_k^\alpha(w)+\frac{\lambda-ir}{\mu_k^\alpha+ir}\,\mathfrak{p}_k^\alpha(w)-\frac{1}{\mu_k^\alpha+ir}\,D_u\bar f(\alpha,u(\alpha))\big(e^{\lambda\cdot}\mathfrak{p}_k^\alpha(w)\big)\right]=0. \quad (8.19)$$

Let F_k^α be the subspace of $\mathfrak{C}_\tau$ spanned by functions of the type $t\to\varphi(t)w$, where $\varphi\in C\big([-\tau,0];\mathbb{C}\big)$ and $w\in E_k^\alpha$. We assume additionally (cf. [113, 139])

(C4) $D_u\bar f(\alpha,u(\alpha))(F_k^\alpha)\subset E_k^\alpha$ for all steady-state solutions $(\alpha,u(\alpha))$ and $k=0,1,2,\dots$.

Remark 8.2.4. The assumption (C4) is required mainly to simplify the computation of the characteristic roots through a reduction to isotypical components of $L^2(\Omega,V^c)$ (see also [137, 138]). One can check that the reaction-diffusion systems with delay of the type considered in [35, 36, 37] satisfy (C4). In the case of a parabolic system of Γ-symmetric PDEs without delay, or the reaction-diffusion logistic equation with delay , (C4) is automatically satisfied (cf. [95]).

Under the assumption (C4), the equation (8.19) can be reduced to

$$\mathfrak{p}_k^\alpha(w)+\frac{\lambda-ir}{\mu_k^\alpha+ir}\mathfrak{p}_k^\alpha(w)-\frac{1}{\mu_k^\alpha+ir}\,D_u\bar f(\alpha,u(\alpha))\big(e^{\lambda\cdot}\mathfrak{p}_k^\alpha(w)\big)=0, \quad (8.20)$$

for $k=0,1,\dots$. The equation (8.20) can be re-written as

$$(\mu_k^\alpha+\lambda)\mathfrak{p}_k^\alpha(w)+D_u\bar f(\alpha,u(\alpha))\big(e^{\lambda\cdot}\mathfrak{p}_k^\alpha(w)\big)=0, \quad k=0,1,\dots . \quad (8.21)$$

8.2.6 Local Bifurcation Invariant and Its Computation

Under the assumptions (C1)—(C4), for any compact subset $X\subset\mathcal{P}$, the system (8.11) leads to a parameterized equivariant coincidence problem of the type (8.2) satisfying (H1)—(H5). Following the construction outlined in Section 8.1.2, given an isolated center (α_o,u_o) with the corresponding characteristic root $i\beta_o$, we associate to (α_o,β_o,u_o) a local bifurcation invariant $\omega(\alpha_o,\beta_o,u_o)\in A_1^t(\Gamma\times S^1)$ (cf. Definition 8.1.4).

To establish an effective computational formula for $\omega(\alpha_o,\beta_o,u_o)$, we need to obtain information about the negative spectrum and the isotypical crossing numbers.

Negative Spectrum

Assume that Γ is a finite group. Suppose that V (resp. V^c) takes the isotypical decomposition (6.6) (resp. (6.7)). Then, it induces the Γ-isotypical decompositions

$$L^2(\Omega;V)=\bigoplus_{i=0}^{r}\mathfrak{V}_i,\quad L^2(\Omega;V^c)=\bigoplus_{j=0}^{s}\mathfrak{U}_j, \tag{8.22}$$

where $\mathfrak{V}_i := L^2(\Omega;V_i)$ (resp. $\mathfrak{U}_j := L^2(\Omega;U_j)$) is modeled on $\mathcal{V}_i$ (resp. $\mathcal{U}_j$).

Consider the operator $P(\alpha_o,x):\mathfrak{B}_{\alpha_o}\to L^2(\Omega;V)$ and let K be the orthogonal projection on its kernel. Then, K is a Γ-equivariant resolvent of $P(\alpha_o,x)$. Put $\widetilde{R}_{\alpha_o} := [P(\alpha_o,x)+K]^{-1}$ and define

$$\mathcal{A} := \operatorname{Id}-\frac{1}{\beta_o}\widetilde{R}_{\alpha_o}\circ D_u\bar{f}(\alpha_o,u_o)-\widetilde{R}_{\alpha_o}K : L^2(\Omega;V)\to L^2(\Omega;V). \tag{8.23}$$

Denote by $\sigma_-(\mathcal{A})$ the set of all negative eigenvalues of the operator $\mathcal{A}$. Since $\mathcal{A}$ is a compact field, the set $\sigma_-(\mathcal{A})$ is finite and each eigenvalue is of finite multiplicity. Thus, for $\mu\in\sigma_-(\mathcal{A})$, define

$$\begin{aligned}E(\mu) &:= \bigcup_{k=1}^{\infty}\ker[\mathcal{A}-\mu\operatorname{Id}]^k,\\ E_i(\mu) &:= \bigcup_{k=1}^{\infty}\ker[\mathcal{A}|_{\mathfrak{V}_i}-\mu\operatorname{Id}|_{\mathfrak{V}_i}]^k,\\ m_i(\mu) &:= \dim E_i(\mu)/\dim\mathcal{V}_i,\end{aligned} \tag{8.24}$$

where the subspace $E(\mu)$ refers to a *generalized eigenspace* of the operator $\mathcal{A}$ and the integer $m_i(\mu)$ will be called the $\mathcal{V}_i$-*multiplicity* of μ.

In all the examples considered in the next section, the condition (R1) from Subsection 6.3.4 is satisfied, as well as the following

(R2)' For each $\mu\in\sigma_-(\mathcal{A})$, there exists a *single* isotypical component $\mathfrak{V}_i$ for $i=i_\mu$ in (8.22), which contains $E(\mu)$ completely.

Therefore, the formula (8.24) of the $\mathcal{V}_i$-multiplicity $m_i(\mu)$ reduces to

$$m_i(\mu)=\begin{cases}\dim E(\mu)/\dim\mathcal{V}_i & i=i_\mu,\\ 0 \quad i\neq i_\mu.\end{cases} \tag{8.25}$$

Crossing Numbers

Put $\widetilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda) := \widetilde{\Delta}^r_{\alpha;u(\alpha)}(\lambda)|_{\mathcal{U}_j}$. For a characteristic root λ of the system (8.11) at the Γ-symmetric steady-state solution $(\alpha, u(\alpha))$, we use the following notations

$$
\begin{aligned}
E_j(\lambda) &:= \bigcup_{k=1}^{\infty} \ker[\widetilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda)]^k, \\
m_j(\lambda) &:= \dim E_j(\lambda)/\dim \mathcal{U}_j,
\end{aligned}
\tag{8.26}
$$

where the subspace $E_j(\lambda)$ is referred to as a *generalized kernel* of the operator $\widetilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda)$ and the integer $m_j(\lambda)$ will be called the *$\mathcal{U}_j$-multiplicity* of the characteristic root λ. Since $\widetilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda)$ is a Fredholm operator of index 0, $m_j(\lambda) < \infty$ for each λ.

Let $(\alpha_o, u_o) \in \mathbb{R} \oplus L^2(\Omega; V)$ be an isolated center with $i\beta_o$ ($\beta_o > 0$) being a corresponding characteristic root as assumed in (C3) from Subsection 8.2.5. Define the set

$$\mathcal{S} = \{\tau + i\beta : 0 < \tau < \delta, \ \ |\beta - \beta_o| < \varepsilon\} \subset \mathbb{C},$$

where $\delta > 0$ and $\varepsilon > 0$ are so small numbers that for all $\tau + i\beta \in \partial\mathcal{S}$ and $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$, $\ker \Delta_{\alpha;u(\alpha)}(\tau + i\beta) \neq \{0\}$ implies $\alpha = \alpha_o$ and $\tau + i\beta = i\beta_o$. Put $\alpha_\pm := \alpha_o \pm \varepsilon$ and denote by $\mathfrak{s}_\pm$ the set of all characteristic roots $\lambda \in \mathcal{S}$ for $\alpha = \alpha_\pm$, i.e.

$$\mathfrak{s}_\pm := \{\lambda \in \mathcal{S} : \ker \Delta_{\alpha_\pm;u(\alpha_\pm)}(\lambda) \neq \{0\}\}.$$

Since $\ker \Delta_{\alpha_\pm;u(\alpha_\pm)}(\lambda) = \ker \widetilde{\Delta}_{\alpha_\pm;u(\alpha_\pm)}(\lambda)$ and $\widetilde{\Delta}_{\alpha_\pm;u(\alpha_\pm)}(\lambda)$ is an analytic function in λ, the sets $\mathfrak{s}_\pm$ are finite.

For $j = 0, 1, 2, \dots, s$, put

$$\mathfrak{t}_j^\pm(\alpha_o, \beta_o, u_o) := \sum_{\lambda \in \mathfrak{s}_\pm} m_j(\lambda), \tag{8.27}$$

(cf. (8.26)).

Definition 8.2.5. The *$\mathcal{U}_j$-isotypical crossing number* of (α_o, β_o, u_o) is defined as

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o, u_o) := \mathfrak{t}_j^-(\alpha_o, \beta_o, u_o) - \mathfrak{t}_j^+(\alpha_o, \beta_o, u_o), \tag{8.28}$$

where $\mathfrak{t}_j^\pm(\alpha_o, \beta_o, u_o)$ are given by (8.27). In the case $l\beta_o$ is also a characteristic root of (8.11) at (α_o, u_o) for some integer $l > 1$, put (cf. [15, 6])

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o, u_o) := \mathfrak{t}_{j,1}(\alpha_o, l\beta_o, u_o).$$

Similar as in Subsection 6.3.2, we have cf. [15]

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o, u_o) = -\operatorname{sign} \frac{d}{d\alpha} w(\alpha)|_{\alpha=\alpha_o} m_j(il\beta_o), \tag{8.29}$$

where $w(\alpha)$ stands for the real part of the characteristic root of (8.11) at $(\alpha, u(\alpha))$.

By (R2'), each $E(i\beta_o)$ is completely contained in a *single* isotypical component $\mathfrak{U}_j$ for some $j = j_{\beta_o}$ in (8.22). Thus,

$$m_j(i\beta_o) = \begin{cases} \dim_{\mathbb{C}} E(i\beta_o)/\dim_{\mathbb{C}} \mathcal{U}_j, & j = j_{\beta_o} \\ 0, & j \neq j_{\beta_o}. \end{cases}$$

Based on (8.25) and (8.29), using further homotopy and multiplicativity properties of the twisted primary degree (cf. Section 4.2), following a similar derivation in Section 6.2, one can establish the following computational formula

$$\omega(\alpha_o, \beta_o, u_o) := \left(\prod_{\mu \in \sigma_-(\mathcal{A})} (\deg_{\mathcal{V}_i})^{m_{i_\mu}(\mu)} \right) \cdot \sum_l \left(-\operatorname{sign} \frac{d}{d\alpha} w(\alpha)|_{\alpha-\alpha_o} m_{j_{\beta_o}}(il\beta_o)\right) \deg_{\mathcal{V}_{j,l}}. \tag{8.30}$$

For simplicity, we will restrict our computations for the first coefficient part of $\omega(\alpha_o, \beta_o, u_o)$ (cf. Subsection 6.3.3), i.e.

$$\omega(\alpha_o, \beta_o, u_o)_1 := \left(\prod_{\mu \in \sigma_-(\mathcal{A})} (\deg_{\mathcal{V}_i})^{m_{i_\mu}(\mu)} \right) \cdot \left(-\operatorname{sign} \frac{d}{d\alpha} w(\alpha)|_{\alpha=\alpha_o} m_{j_{\beta_o}}(i\beta_o)\right) \deg_{\mathcal{V}_{j,1}}. \tag{8.31}$$

Combining the concept of the dominating orbit types with Theorem 8.1.5, one can easily establish a similar result stated in Theorem 6.1.9

Theorem 8.2.6. *Suppose that the system (8.11) satisfies the assumption (C1) and (C4), and suppose that* (α_o, u_o) *is a* Γ*-symmetric steady-state solution to (8.11) (cf. Definition 8.2.1) satisfying (C2)—(C3),* $\omega(\alpha_o, \beta_o, u_o)$ *is given by (8.10) (with* $\lambda_o = (\alpha_o, \beta_o)$*,* $\mathfrak{F}_\varsigma$ *defined by (8.9),* $U(r)$ *by (8.7) and* ς *satisfying (8.8)). Assume (cf. (8.30))* $\omega(\alpha_o, \beta_o, u_o) \neq 0$*, i.e.*

$$\omega(\alpha_o, \beta_o, u_o) = \sum_{(H)} n_H(H) \quad \text{and} \quad n_{H_o} \neq 0 \tag{8.32}$$

for some $(H_o) \in \Phi_1(G)$.

(i) Then, there exists a branch of non-trivial solutions to (8.11) with symmetries at least H_o *(considered in the space* $\mathbb{F}$*) bifurcating from the point* (α_o, u_o) *(with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$*).*

(ii) If, in addition, (H_o) *is a dominating orbit type in* $\mathbb{F}$*, then there exist at least* $|G/H_o|_{S^1}$ *different branches of periodic solutions to the equation (8.11) bifurcating from* (α_o, u_o)*. Moreover, for each* (α, β, u) *belonging to these branches of (non-trivial) solutions one has* $(G_u) = (H_o)$ *(considered in the space* $\mathbb{F}$*).*

Remark 8.2.7. The setting presented in this section for the functional parabolic differential equations can be extended to a more general situation when $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_1 and Γ_2 are finite groups acting orthogonally on V' and V respectively, and $\Omega \subset V'$ is an open bounded Γ_1-invariant set with C^2-smooth boundary. Then, the Banach space $L^2(\mathbb{R} \times \overline{\Omega}; V)$ is again an isometric Γ-representation with the Γ-action given by

$$(\gamma u)(t, x) = \gamma_2(u(t, \gamma_1 x)), \quad \gamma = (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2.$$

8.3 Symmetric System of Hutchinson Model in Population Dynamics

8.3.1 A Hutchinson Model of an n Species Ecosystem

We start with the standard model for the dynamics of a simple (single) population[*] in terms of its density — the Verhulst equation (cf. [93, 84])

$$\dot{v} = \alpha v \left(1 - \frac{v}{K}\right),$$

which is based on the idea that the population grows exponentially at low densities and saturates towards the carrying capacity K (of resources) at high densities. By taking into account a delayed response to the remaining resources, the Hutchinson's model (of a single species) is obtained

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[*] For population ecology background, we refer to [93, 162, 66].

$$\dot{v}(t) = \alpha v(t) \left(1 - \frac{v(t-\tau)}{K}\right), \tag{8.33}$$

where $\tau > 0$ is a presumed delay constant and α refers to the intrinsic growth rate.

Now, we consider an ecosystem composed of n species interacting with each other (according to a certain symmetry) by competing (or cooperating) over shared resources such as food and habitats, while maintaining a self-inhibiting nature (meaning self-limiting in respond to rare resources and self-reproducing to abundant resources). A mathematical treatment for such a community model was developed by Levins in [126], where one attaches a loop diagram in order to carry out a loop analysis for this community type situation (cf. Figure8.1).

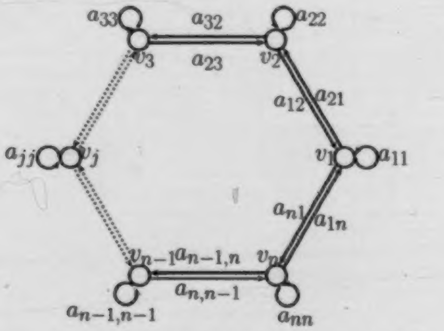

Fig. 8.1. System with dihedral symmetries

In Figure 8.1, a_{jj} describes the self-inhibiting nature of the j-th species, and $a_{ij} < 0$ (resp. $a_{ij} > 0$) is the competing (resp. cooperating) coefficient between species i and j. Also, observe that $a_{ij} = a_{ji}$. We introduce

$$C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{8.34}$$

and call it the *community matrix.* We describe this community ecosystem by the following equations,

$$\dot{v}(t) = \alpha C v(t) \cdot \left(1 - \frac{v(t-\tau)}{K}\right), \tag{8.35}$$

where '$\cdot$' is the component-wise multiplication $u \cdot v := [u_1 v_1, \dots, u_n v_n]^T$ for $u = [u_1, \dots, u_n]^T$ and $v = [v_1, \dots, v_n]^T$.

By applying the standard transformation

$$v(t) = K(1 + u(t)), \tag{8.36}$$

to the system (8.35), one obtains the equivalent system

$$\dot{u}(t) = -\alpha\, Cu(t-\tau) \cdot [1 + u(t)], \tag{8.37}$$

where $u(t) = \frac{v(t)}{K} - 1$ is, in fact, a population saturation index with respect to the available resources.

Finally, to study the system (8.37) in a heterogeneous environment, we add to (8.37) a spatial diffusion term, which leads to the following reaction-diffusion equations

$$\frac{\partial}{\partial t} u(x,t) = d \frac{\partial^2}{\partial x^2} u(x,t) - \alpha\, Cu(x, t-1)[1 + u(x,t)], \tag{8.38}$$

where $d > 0$ is a spatial diffusion coefficient.

8.3.2 A Symmetric System of the Hutchinson Model

We consider a symmetric system of n species Hutchinson model of the form (8.38) (for $t > 0$ and $x \in (0, \pi)$)

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = d \frac{\partial^2}{\partial x^2} u(x,t) - \alpha\, Cu(x, t-1) \cdot [1 + u(x,t)], \\ \frac{\partial}{\partial x} u(x,t) = 0, \quad x = 0, \pi, \end{cases} \tag{8.39}$$

where $u : [0, \pi] \times \mathbb{R} \to \mathbb{R}^n$ is a population saturation index (cf. (8.37)), '$\cdot$' is the component-wise multiplication, $d > 0$ is a spatial diffusion coefficient and $\alpha \neq 0$ is the intrinsic growth rate (cf. (8.33)), which is considered as a bifurcation parameter, and C is a (symmetric) community matrix describing the interaction among the species.

Assume that

(A1) The geometrical configuration described by the system (8.39) has a symmetry group Γ. The group Γ permutes the vertices of the related polygon or polyhedron, which means it acts on $\mathbb{R}^n$ by permuting the coordinates of the vectors $x \in \mathbb{R}^n$. The matrix C commutes with this Γ-action and $0 \notin \sigma(C)$.

Under the assumption (A1), the space $V := \mathbb{R}^n$ becomes an orthogonal Γ-representation and the condition (C1) from Subsection 8.2.1 is satisfied by the system (8.39).

8.3.3 Characteristic Equation and Isolated Centers

At a Γ-symmetric steady-state solution $(\alpha, 0)$, the system (8.39) has the linearization

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = d \frac{\partial^2}{\partial x^2} u(x,t) - \alpha\, C\, u(x, t-1), \\ \frac{\partial}{\partial x} u(x,t) = 0, \quad x = 0, \pi. \end{cases} \tag{8.40}$$

Since the matrix C is symmetric, it is completely diagonalizable with respect to a basis composed of its eigenvectors. Consider the spectrum $\sigma(C) = \{\xi_1, \xi_2, \dots, \xi_q\}$ of the matrix C and denote by $E(\xi_k) \subset V$ the eigenspace of ξ_k. Then,

$$L^2([0,\pi]; V) = \bigoplus_{k=1}^{q} L^2([0,\pi]; E(\xi_k)), \tag{8.41}$$

and $w \in L^2([0,\pi]; V)$ can be represented as $w(x) = \sum_k w_k(x)$, where $w_k \in L^2([0,\pi]; E(\xi_k))$. Similarly, we have

$$L^2([0,\pi]; V^c) = \bigoplus_{k=1}^{q} L^2([0,\pi]; E^c(\xi_k)), \tag{8.42}$$

where $E^c(\xi_k)$ denotes the complexification of the eigenspace $\mathfrak{E}(\xi_k)$.

Notice that $(\alpha_o, 0)$ is a Γ-symmetric steady-state solution to (8.39) for all $\alpha \neq 0$. Thus, we can take the set $\{(\alpha, \beta, 0) \ : \ \alpha \neq 0\}$, for the manifold $M \subset \mathcal{P} \times \mathbb{E}^G$ described in Subsection 8.1.2. Moreover, $(\alpha_o, 0)$ is nonsingular if $0 \notin \sigma(\mathcal{L}_{\alpha_o})$, where

$$\mathcal{L}_{\alpha_o} := d\frac{\partial^2}{\partial x^2} - \alpha_o C : H_0^2([0,\pi]; V) \to L^2([0,\pi); V]$$

with $H_0^2([0,\pi];V)$ being the subspace of $H^2([0,\pi];V)$ consisting of functions u satisfying $u(0) = u(\pi) = 0$. One can easily verify that if

$$-\frac{\alpha_o \xi_k}{d} \neq m^2 \quad \text{for all} \quad k = 1, 2, \dots, q, \quad \text{and} \quad m = 0, 1, 2, \dots,$$

then $(\alpha_o, 0)$ is a nonsingular Γ-symmetric steady-state solution, i.e. $(\alpha_o, 0)$ satisfies the condition (C2) from Subsection 8.1.2.

A number $\lambda \in \mathbb{C}$ is a characteristic root of the system (8.39) at a Γ-symmetric steady-state solution $(\alpha, 0) \in \mathbb{R} \oplus V$ if there exists a nonzero function $v \in L^2([0,\pi]; V^c)$ such that

$$\triangle_\alpha(\lambda)v(x) := \lambda v(x) - d\frac{\partial^2}{\partial x^2}v(x) + \alpha e^{-\lambda} C\, v(x) = 0, \tag{8.43}$$

where we put $\triangle_\alpha := \triangle_{\alpha;0}$ (cf. (8.17)).

By using the decomposition (8.42), v can be written as $v(x) = \sum\limits_k v_k(x)$, for $v_k(x) \in E(\xi_k)$. Consequently, (8.43) yields

$$\triangle_\alpha(\lambda)v(x) = \sum_k \left(\lambda v_k(x) - d\frac{\partial^2}{\partial x^2}v_k(x) + \alpha e^{-\lambda}\xi_k\, v_k(x) \right) = 0. \tag{8.44}$$

Next, by using the point spectrum $\{\zeta_m := dm^2\}_{m=0}^{\infty}$ of the (scalar-valued) Laplace operator $L := -d\frac{\partial^2}{\partial x^2}$ and the corresponding eigenspaces $E(\zeta_m)$, we can write $v_k(x) = \sum\limits_m v_{k,m}(x)$, for $v_{k,m} \in E(\zeta_m)$, thus

$$\triangle_\alpha(\lambda)v(x) = \sum_{k,\, m} \left(\lambda v_{k,m}(x) + dm^2 v_{k,m}(x) + \alpha e^{-\lambda}\xi_k\, v_{k,m}(x)\right) = 0. \tag{8.45}$$

Therefore, one obtains that $\lambda \in \mathbb{C}$ is a characteristic root of (8.40) at the Γ-symmetric steady-state solution $(\alpha, 0)$, if

$$\lambda + dm^2 + \alpha\,\xi_k e^{-\lambda} = 0, \qquad \text{for} \quad k = 1, \dots, q \text{ and } m = 0, 1, \dots \tag{8.46}$$

8.3.4 Computations for the Local Bifurcation $\Gamma \times S^1$-Invariant

In order to find the values α_o for which the condition (C3) from Subsection 8.2.5 holds, we need to find purely imaginary roots $\lambda = i\beta$ ($\beta > 0$) of (8.46). Assume that $(\alpha, 0)$ is a nonsingular steady-state solution to (8.39) (in particular, $\alpha \neq 0$).

- **Computation for purely imaginary roots $\lambda = i\beta$ $(\beta > 0)$**

By substituting $\lambda = i\beta$ into (8.46), we obtain

$$\begin{cases} dm^2 + \alpha\xi_k \cos\beta = 0, \\ \beta - \alpha\xi_k \sin\beta = 0. \end{cases} \quad \text{for} \quad k = 1, \dots, q. \tag{8.47}$$

In the case $m = 0$, we have

$$\begin{cases} \beta := \beta_{\nu,0,k} = \frac{\pi}{2} + \nu\pi, \\ \alpha := \alpha_{\nu,0,k} = (-1)^\nu \frac{\beta}{\xi_k}, \end{cases}$$

for $k = 1, \dots, q$ and $\nu = 0, 1, \dots$. Consequently,

$$\operatorname{sign} \alpha_{\nu,0,k} = (-1)^\nu \operatorname{sign} \xi_k. \tag{8.48}$$

In the case $m \neq 0$ (thus $\cos\beta \neq 0$ by the first equation in (8.47)), we obtain

$$\tan\beta = -\frac{\beta}{dm^2}, \tag{8.49}$$

$$\alpha = -\frac{dm^2}{\xi_k \cos\beta}, \tag{8.50}$$

The equation (8.49) has infinitely many positive solutions, which will be denoted by $\{\beta_{\nu,m,k}\}_{\nu=1}^{\infty}$ (see Figure 8.2). The corresponding solution α of (8.50) will be denoted by $\alpha_{\nu,m,k}$.

Also, we notice that $\operatorname{sign} \cos\beta_{\nu,m,k} = (-1)^\nu$, thus by (8.50),

$$\operatorname{sign} \alpha_{\nu,m,k} = (-1)^{\nu+1} \operatorname{sign} \xi_k. \tag{8.51}$$

- **Computation for $\operatorname{sign} \frac{d}{d\alpha} w(\alpha)|_{\alpha=\alpha_{\nu,m,k}}$**

Put $\alpha_o := \alpha_{\nu,m,k}$ and $\beta_o := \beta_{\nu,m,k}$. In order to determine the value of the crossing number $t_{j,1}(\alpha_o, \beta_o, 0)$, we need to compute $\frac{d}{d\alpha} w(\alpha)|_{\alpha=\alpha_o}$ by implicit differentiation.

By substituting $\lambda = w + iv$ into (8.46),

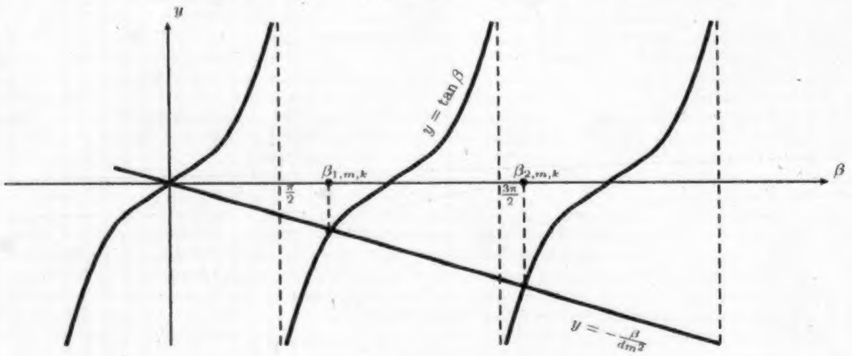

Fig. 8.2. Purely imaginary roots of the characteristic equation.

$$\begin{cases} w + dm^2 + \alpha\xi_k e^{-w}\cos v = 0, \\ v - \alpha\xi_k e^{-w}\sin v = 0, \end{cases} \tag{8.52}$$

then, differentiating (8.52) with respect to α, we obtain

$$\begin{cases} \frac{dw}{d\alpha} - \alpha\xi_k e^{-w}(\frac{dw}{d\alpha}\cos v + \frac{dv}{d\alpha}\sin v) = -\xi_k e^{-w}\cos v, \\ \frac{dv}{d\alpha} + \alpha\xi_k e^{-w}(\frac{dw}{d\alpha}\sin v - \frac{dv}{d\alpha}\cos v) = \xi_k e^{-w}\sin v, \end{cases} \tag{8.53}$$

which is equivalent to

$$\begin{cases} \frac{dw}{d\alpha}(\alpha\xi_k e^{-w} - \cos v) + \frac{dv}{d\alpha}\sin v = \xi_k e^{-w}, \\ \frac{dw}{d\alpha}\sin v + \frac{dv}{d\alpha}(\cos v - \alpha\xi_k e^{-w}) = 0. \end{cases} \tag{8.54}$$

Thus, we obtain

$$\frac{dw}{d\alpha} = -\frac{\xi_k e^{-w}(\cos v - \alpha\xi_k e^{-w})}{\alpha^2\xi_k^2 e^{-2w} - 2\alpha\xi_k e^{-w}\cos v + 1}. \tag{8.55}$$

By substituting $\alpha = \alpha_o$, $w = 0$ and $v = \beta_o$, we have

$$\begin{aligned} \frac{dw}{d\alpha}\Big|_{\alpha=\alpha_o} &= -\frac{\xi_k(\cos\beta_o - \alpha_o\xi_k)}{\alpha_o^2\xi_k^2 - 2\alpha_o\xi_k\cos\beta_o + 1} \\ &= -\frac{\xi_k\cos\beta_o - \alpha_o\xi_k^2}{\alpha_o^2\xi_k^2 - 2\alpha_o\xi_k\cos\beta_o + 1}. \end{aligned}$$

Replacing $\xi_k\cos\beta_o$ with $-\frac{dm^2}{\alpha_o}$ in the last equality (cf. (8.50)), we obtain

$$\frac{dw}{d\alpha}\Big|_{\alpha=\alpha_o} = \frac{1}{\alpha_o}\cdot\frac{dm^2 + \alpha_o^2\xi_k^2}{\alpha_o^2\xi_k^2 + 2dm^2 + 1}.$$

Consequently,

$$\operatorname{sign} \frac{dw}{d\alpha}|_{\alpha=\alpha_o} = \operatorname{sign} \alpha_o.$$

Hence, by (8.48) and (8.51), we obtain

$$\operatorname{sign} \frac{dw}{d\alpha}|_{\alpha=\alpha_o=\alpha_{\nu,m,k}} = \begin{cases} (-1)^{\nu} \operatorname{sign} \xi_k, & \text{if} \quad m=0, \\ (-1)^{\nu+1} \operatorname{sign} \xi_k & \text{if} \quad m=1,2,\dots \end{cases} \tag{8.56}$$

Therefore, combining (8.56) with (8.29), we have for $m \neq 0$*

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) = \begin{cases} (-1)^{\nu} \operatorname{sign} \xi_k \dim_{\mathbb{C}} E^c(i\beta_o)/\dim_{\mathbb{C}} \mathcal{U}_j, & j = j_{\beta_o} \\ 0, \quad j \neq j_{\beta_o}. \end{cases} \tag{8.57}$$

8.4 Usage of Maple© Package and Computational Results

In this section, assuming the conditions (C1)–(C4) to be satisfied by the system (8.39), we prepare the input data for using the Maple© routines. The quantative results will be presented in Appendix A4.3, for Γ being the dihedral group D_3 and the tetrahedral group A_4.

Recall that (cf. (8.31))

$$\omega(\alpha_o, \beta_o, 0)_1 = \omega_\Gamma \cdot \omega_G,$$

where

$$\omega_\Gamma = \prod_{\mu \in \sigma_-(\mathcal{A})} (\deg_{\mathcal{V}_i})^{m_{i,\mu}(\mu)},$$

and

$$\omega_G = (-\operatorname{sign} \frac{d}{d\alpha} w(\alpha)|_{\alpha=\alpha_o} m_{j_{\beta_o}}(i\beta_o)) \deg_{\mathcal{V}_{j,1}},$$

with $\mathcal{A}$ being defined for $(\alpha_o, \beta_o) = (\alpha_{\nu,m,k}, \beta_{\nu,m,k})$ (cf. Subsection 8.2.6).

By formula (8.25), we have

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* Throughout the rest of this section, we carry out the computation of the local $\Gamma \times S^1$-invariant $\omega(\alpha_o, \beta_o, u_o)_1 = \omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1$ for $m \neq 0$. In the case $m = 0$, one only needs to change the formula for $\operatorname{sign} \frac{dw}{d\alpha}|_{\alpha=\alpha_o}$ according to (8.56).

$$\omega_\Gamma = \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{\sum\limits_{\mu\in\sigma_-(\mathcal{A})} m_i(\mu)} \tag{8.58}$$

Since $\left(\deg_{\mathcal{V}_i}\right)^2 = (\Gamma)$ for $i = 0, 1, \dots, r$, we can associate with $\sigma_-(\Lambda)$ the sequence $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r)$ defined by

$$\varepsilon_i := \sum_{\mu\in\sigma_-(\mathcal{A})} m_i(\mu) \pmod 2, \quad i = 0, 1, \cdots, r.$$

Then, the formula (8.58) can be reduced to

$$\omega_\Gamma = \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{\varepsilon_i}.$$

Clearly, the sequence $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}$ permits only possibly finitely many different values.

By formula (8.57),

$$\omega_G = (-1)^\nu \dim_{\mathbb{C}} E^c(i\beta_{\nu,m,k})/\dim_{\mathbb{C}} \mathcal{U}_{j_{\beta_{\nu,m,k}}}\ \deg_{\mathcal{V}_{j_{\beta_{\nu,m,k}},1}}.$$

We will use the notation $\mathfrak{m}_{j_{\beta_{\nu,m,k}}} := \dim_{\mathbb{C}} E^c(i\beta_{\nu,m,k})/\dim_{\mathbb{C}} \mathcal{U}_{j_{\beta_{\nu,m,k}}}$, which stands for the $\mathcal{U}_j$-*multiplicity* of $i\beta_{\nu,m,k}$. Thus $\mathfrak{m}_{j_{\beta_{\nu,m,k}}}$ also permits only possibly finitely many different values.

Therefore, we have the following formula for the first coefficients of the local bifurcation invariant

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 = (-1)^\nu \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{\varepsilon_i} \cdot \mathfrak{m}_{j_{\beta_{\nu,m,k}}} \cdot \deg_{\mathcal{V}_{j_{\beta_{\nu,m,k}},1}} \tag{8.59}$$

The input data for the computation of the local invariant thus consists of two finite sequences:

$$\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}, \quad \{\mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_r\},$$

which are forwarded to the following command from the Maple© package

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 := (-1)^\nu \texttt{showdegree}\,[\Gamma](\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r, \mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_r).$$

Remark and Notation 8.4.1 Given $\xi_o \in \sigma(C)$ and assuming (R2)' to be satisfied, in what follows we will use the notation ξ_o^i to indicate that $\widetilde{E}(\xi_o) \subset V_i$ and ${}^j\xi_o$, when $E(\xi_o) \subset U_j$ (here we consider the matrix C acting on V^c). In such a case we will also write ${}^j\xi_o^i$. Since the value of $\mathfrak{m}_{j_{\beta_{\nu,m,k}}}$, by the condition (R), is equal to the $\mathcal{U}_{j_{\beta_{\nu,m,k}}}$-multiplicity $\dim_{\mathbb{C}}(E^c(\xi_k) \cap U_{j_{\beta_{\nu,m,k}}})/\dim_{\mathbb{C}} \mathcal{U}_{j_{\beta_{\nu,m,k}}}$, of the eigenvalue ${}^{j_{\beta_{\nu,m,k}}}\xi_k$ of the complexified matrix C, and $E(i\beta_{\nu,m,k}) \subset \mathfrak{U}_{j_{\beta_{\nu,m,k}}}$, it is convenient to present our quantitative results in a form of a matrix

${}^j\xi_o$	$\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_l}$	$\omega(\lambda_o)_1$	# Branches

where we only list $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_l}\} \subset \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}$ for those ε_{i_t}, which can realize the value 1.

Remark 8.4.2. Although we are dealing with *infinitely* many isolated centers

$$(\alpha_o, \beta_o, 0) \in \{(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)\}_{\nu,m,k},$$

only *finitely* many different values of $\omega(\alpha_o, \beta_o, 0)_1$ may occur, which is related to the fact that the value of $\omega(\alpha_o, \beta_o, 0)_1$ is determined by only possibly *finitely* many different choices of the values of the two sequences $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}$ and $\{\mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_r\}$.

9

Existence of Periodic Solutions to Symmetric Lotka-Volterra Type Systems

In the previous chapters (cf. Chapters 6—8), the primary equivariant degree method was adapted to study the Hopf bifurcation problem in the symmetric (neutral) functional differential equations and parabolic partial differential equations. In this chapter, we extend the scope of the applications of the primary degree to the existence problem of nonstationary periodic solutions in a symmetric system of functional differential equations. In particular, we discuss the existence of periodic solutions to a symmetric Lotka-Volterra system with delays, which falls out of the category of symmetric variational problems. It should be pointed out that while a large variety of effective topological methods and techniques can be applied to symmetric variational problems (cf. [19] and references therein), in the case of symmetric *non-variational* problems, there are only few topological methods which are traditionally used. Unfortunately, some of those methods (eg. Leray-Schauder degree) are ineffective for detecting nonstationary periodic solutions.

The Lotka-Volterra equation, being the simplest model of predator-prey interactions, plays an important role in the population dynamics. In this chapter, we are interested in exploring the symmetric aspect of such model by considering a symmetrically configured community of N-species competing/cooperating for the shared resources, described by a symmetric Lotka-Volterra type system (cf. (9.1)). Following the original idea in [90], we introduce additional (homotopy) parameters to the system and establish *a priori* bounds for the parametrized systems (9.2_α) and $(9.2_{\alpha\rho})$. Based on *a priori* bounds, using a standard homotopy argument, we define a topological invariant '$\boxdot$' (cf. Definition 9.1.1), which contains information about the existence of multiple nonstationary periodic solutions of (9.1).

Although hardly anything in biological systems is exactly symmetric, when dealing with models of limited accuracy, one can place the considered models in a symmetric setting, which allows us to explore and better understand certain symmetric impact on the dynamics of such systems. Being able to establish the existence of multiple periodic solutions in such a system provides us with a third eye in observing the complexity of its dynamics, including explaining the appearance of patterns in synchronized fluctuations of populations.

This chapter is organized as follows. In Section 9.1, we present a general framework for studying the existence of nonstationary periodic solutions to a Γ-symmetric system of delayed differential equations. Based on *a priori* bounds assumed for parametrized (by additional parameters) systems (which are applied to construct an appropriate admissible homotopy), we define a $\Gamma \times S^1$-equivariant topological invariant '⧈' containing structural information about the solution set of our considered system. The existence and multiplicity results can be easily extracted from '⧈'. Computational formula is derived based on the multiplicativity and homotopy property of the twisted primary degree. In Section 9.2, we apply the general framework to a Γ-symmetric Lotka-Volterra system. Especially, the required *a priori* bounds are established step by step using specific properties of the parametrized systems. Consequently, the equivariant topological invariant is associated to the symmetric Lotka-Volterra system and evaluated according to the computational formula discussed previously. In Section 9.3, we briefly explain the usage of the Maple© routines. The sample computations are included in Appendix A4.4, for $\Gamma = Q_8, D_8, S_4$.

9.1 Existence Problem in Symmetric Delayed Differential Equations

We present a general framework for studying the existence of nonstationary periodic solutions to a system of symmetric delayed differential equations. Throughout this section, assume that Γ is a compact Lie group and V is an orthogonal Γ-representation.

9.1.1 Statement of the Problem

For a given constant $\tau > 0$, consider the Banach space $C_{V,\tau}$ defined by (6.1) equipped with the norm given by (6.2), which is a natural isometric Banach representation of Γ (cf. (6.3)). For a continuous function $x : \mathbb{R} \to V$ and $t \in \mathbb{R}$, define $x_t \in C_{V,\tau}$ by (6.4).

Assume that

(A1) $\mathscr{A} : C_{V,\tau} \to V$ is a bounded Γ-equivariant linear operator. Moreover, $\mathscr{B} := \mathscr{A}|_V$ is a linear isomorphism from V to V.

(A2) $\mathscr{R} : C_{V,\tau} \to V$ is a continuously differentiable Γ-equivariant map, such that $\mathscr{R}(0) = 0$ and $D\mathscr{R}(0) = 0$.

We are interested in finding a continuously differentiable function $u : \mathbb{R} \to V$ satisfying the following autonomous functional differential equation

$$\begin{cases} \dot{u}(t) = \mathscr{A}(u_t) + \mathscr{R}(u_t), \\ u(0) = u(p), \end{cases} \tag{9.1}$$

where $p > 0$ is the unknown period of u.

9.1.2 Normalization of Period

By normalization of the period in (9.1), we understand the following change of variable $x(t) = u(\lambda t)$, where $\lambda = \frac{p}{2\pi}$ is considered to be a *new* parameter. We obtain the following equation, which is equivalent to (9.1)

$$\begin{cases} \dot{x}(t) = \lambda \left[\mathscr{A}(x_{t,\lambda}) + \mathscr{R}(x_{t,\lambda})\right], \\ x(0) = x(2\pi), \end{cases} \tag{9.2}$$

where $x : \mathbb{R} \to V$, $x_{t,\lambda} \in C_{V,\tau}$ is defined by $x_{t,\lambda}(\theta) := x\left(t + \frac{\theta}{\lambda}\right)$, $\theta \in [-\tau, 0]$.

9.1.3 Setting in Functional Spaces

By using the standard identification of $\mathbb{R}/2\pi\mathbb{Z}$ with S^1, we consider the first Sobolev space of 2π-periodic functions

$$\mathbb{H} := H^1(S^1; V), \tag{9.3}$$

which is equipped with the inner product

$$\langle u, v\rangle_{H^1} := \int_0^{2\pi} \dot{u}(t)\dot{v}(t)dt + \int_0^{2\pi} u(t)v(t)dt, \quad u, v \in \mathbb{H},$$

and the induced norm will be denoted by $\|\cdot\|_{H^1}$. Notice that $\mathbb{H}$ is a natural isometric Hilbert G-representation for $G = \Gamma \times S^1$ (cf. (6.21)).

The existence result for the equation (9.1) under the assumptions (A1) and (A2), can be obtained by the means the twisted primary G-equivariant degree using the standard homotopy argument and *a priori* bounds for the following two equations

$$\begin{cases} \dot{x}(t) = \alpha\lambda[\mathcal{A}(x_{t,\lambda}) + \mathcal{R}(x_{t,\lambda})] \\ x(0) = x(2\pi), \end{cases} \tag{9.2$_\alpha$}$$

and

$$\begin{cases} \dot{x}(t) = \alpha\lambda[\mathcal{A}(x_{t,\lambda}) + \rho\mathcal{R}(x_{t,\lambda})] \\ x(0) = x(2\pi); \end{cases} \tag{9.2$_{\alpha\rho}$}$$

where $\rho \in [0,1]$, $\alpha \in (0,1]$ and $\lambda \in [\lambda_1, \lambda_2]$ for fixed constants $0 < \lambda_1 < \lambda_2$.

More precisely, we rewrite the equation $(9.2_{\alpha\rho})$ in functional spaces as

$$Lx = \alpha\lambda[N_{\mathcal{A}}(\lambda, j(x)) + \rho N_{\mathcal{R}}(\lambda, j(x))], \tag{9.4}$$

where L, j are defined by (6.16)—(6.17) and

$$N_{\mathcal{A}} : \mathbb{R}_+ \times C(S^1; V) \to L^2(S^1; V), \quad N_{\mathcal{A}}(\lambda, x)(t) = \mathcal{A}(x_{t,\lambda}), \tag{9.5}$$

$$N_{\mathcal{R}} : \mathbb{R}_+ \times C(S^1; V) \to L^2(S^1; V), \quad N_{\mathcal{R}}(\lambda, x)(t) = \mathcal{R}(x_{t,\lambda}). \tag{9.6}$$

Using the (finite-dimensional) operator $K : \mathbb{H} \to L^2(S^1; V)$ defined by (6.18), the equation $(9.2_{\alpha\rho})$ is equivalent to

$$x - \alpha\lambda(L+K)^{-1}[N_{\mathcal{A}}(\lambda, j(x)) + \rho N_{\mathcal{R}}(\lambda, j(x)) + Kx] = 0, \quad x \in \mathbb{H}. \tag{9.7}$$

9.1.4 *A Priori* Bounds

To define a G-equivariant topological invariant for $(9.2_{\alpha\rho})$ which is valid for any $\rho \in [0,1]$ using admissible homotopy argument, we need to establish the *a priori* bounds for (9.2_α) and $(9.2_{\alpha\rho})$. As it turns out, the *a priori* bounds are closely related to the properties of $\mathcal{A}$ and $\mathcal{R}$. In this general setting, we only describe the required properties of the *a priori* bounds (cf. (P1)—(P5)), and define the region of the admissible homotopy based on the *a priori* bounds.

We assume

(P0) There exists an open G-invariant set $\mathcal{C} \subset \mathbb{H}$ such that $0 \in \mathcal{C}$ and for every solution $x \in \mathcal{C}$ to (9.2_α), we have

$$\int_0^{2\pi} x(t)dt = 0.$$

We also assume that the following *a priori* bounds for (9.2_α) and $(9.2_{\alpha\rho})$.

(P1) There exists $\alpha_o \in (0,1)$ such that for all $0 \leq \alpha \leq \alpha_o$, $\rho \in [0,1]$ and $\lambda \in [\lambda_1, \lambda_2]$ the system ($9.2_{\alpha\rho}$) has no nontrivial solution in $\mathcal{C}$.

(P2) There exist an open bounded G-invariant set $\widetilde{\mathcal{U}} \subset \mathcal{C}$ such that for a small $\varepsilon > 0$ and

$$\mathcal{U} := \{x \in \mathbb{H} : \operatorname{dist}(x, \widetilde{\mathcal{U}}) < \varepsilon\},$$

the following inclusion is satisfied

$$0 \in \widetilde{\mathcal{U}} \subset \overline{\mathcal{U}} \subset \mathcal{C}.$$

Moreover, every nontrivial solution in $\mathcal{C}$ to ($9.2_{\alpha\rho}$) belongs to $\widetilde{\mathcal{U}}$, for $\alpha \in (0,1]$ and $\lambda \in [\lambda_1, \lambda_2]$.

Since we do not specify here exactly what is the set $\widetilde{\mathcal{U}}$, we should explain that we expect that it is of "good" type, for example a star-shaped open set around the origin in $\mathbb{H}$.

In order to control the solutions near the origin, we assume that

(P3) There exists $m_1 > 0$ such that ($9.2_{\alpha\rho}$) for $\alpha = 1$ and $\rho \in [0,1]$, has no nontrivial solution in $\overline{B} := \{x \in \mathbb{H} : \|x\|_{H^1} \leq m_1\} \subset \widetilde{\mathcal{U}}$.

Finally, we also need

(P4) The system ($9.2_{\alpha\rho}$), for $\alpha = 1$ and $\rho = 0$, does not have nontrivial solutions in $\mathbb{H}$.

(P5) For $\lambda = \lambda_i$, $i = 1, 2$, the system ($9.2_{\alpha\rho}$) has no nontrivial solution in $\mathcal{U}$.

Let λ_1, λ_2 be given by (P5) and the sets $\mathcal{U}$, B be given by (P2), (P3) respectively. Define

$$\Omega_{\lambda_1,\lambda_2} := \{(\lambda, x) : \lambda_1 < \lambda < \lambda_2, \ \ x \in \mathcal{U} \setminus \overline{B}\}. \tag{9.8}$$

9.1.5 Control Function β

Choose α_1 with $0 < \alpha_1 < \alpha_o$, to be sufficiently small and take a continuous function $\xi : [0, \infty) \to [\alpha_1, 1]$ such that (see Figure 9.1)

$$\xi(t) = \begin{cases} 1, & \text{if } \ t = 0, \\ \text{strictly decreasing} & \text{if } \ 0 \leq t \leq \varepsilon, \\ \alpha_1, & \text{if } \ t > \varepsilon. \end{cases} \tag{9.9}$$

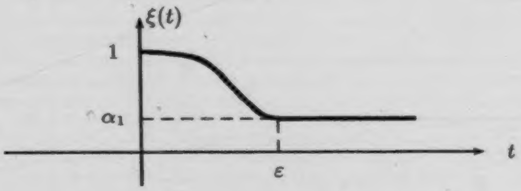

Fig. 9.1. Bump function $\xi : [0, +\infty) \to [\alpha_1, 1]$

Define $\beta : \mathbb{H} \to \mathbb{R}_+$ by

$$\beta(x) = \xi(\operatorname{dist}(x, \widetilde{\mathcal{U}})). \tag{9.10}$$

Next replace α in $(9.2_{\alpha\rho})$ by $\beta(x)$, i.e. consider the equation

$$\begin{cases} \dot{x}(t) = \beta(x)\lambda[\mathcal{A}(x_{t,\lambda}) + \rho\mathcal{R}(x_{t,\lambda})] \\ x(0) = x(2\pi). \end{cases} \tag{$9.2_{\beta\rho}$}$$

Notice that for $\rho = 1$, $(9.2_{\beta\rho})$ has exactly the same solution set in $\Omega_{\lambda_1,\lambda_2}$ as (9.2). The considered sets and the function β are illustrated on Figure 9.2.

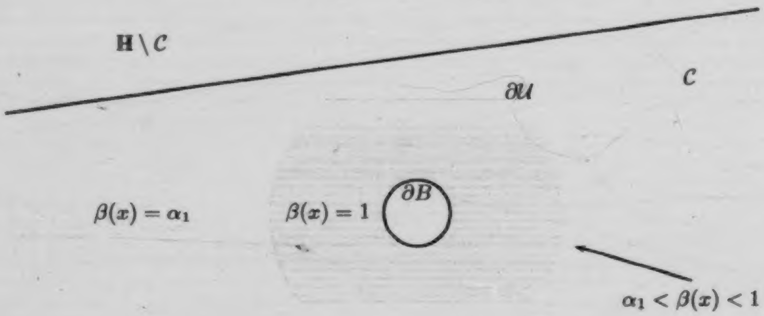
$\mathbb{H} \setminus \mathcal{C}$
$\partial\mathcal{U}$
$\mathcal{C}$
$\beta(x) = \alpha_1$
$\beta(x) = 1$
∂B
$\alpha_1 < \beta(x) < 1$

Fig. 9.2. The sets $\mathcal{U} \setminus \overline{B}$, $\partial\mathcal{U}$ and ∂B.

9.1.6 Admissible Homotopy

Define for $\rho \in [0, 1]$,

$$\mathfrak{F}_\rho(\lambda, x) := x - \beta(x)\lambda(L+K)^{-1}[N_{\mathfrak{A}}(\lambda, j(x)) + \rho N_{\mathfrak{K}}(\lambda, j(x)) + Kx], \quad (9.11)$$

which is an $\Omega_{\lambda_1,\lambda_2}$-admissible homotopy by (P1)—(P5). Indeed, observe that for $x \in \partial\mathcal{U}$, $\beta(x) = \alpha_1 < \alpha_o$, thus by (P1), $\mathfrak{F}_\rho(\lambda, x) \neq 0$ for $\lambda \in [\lambda_1, \lambda_2]$. On the other hand, by (P3), $\mathfrak{F}_\rho(\lambda, x) \neq 0$ for $x \in \partial\overline{B}$. Therefore, one only needs to show that for $\lambda = \lambda_i$, $i = 1, 2$, $\mathfrak{F}_\rho(\lambda_i, x) \neq 0$ for $x \in \mathcal{U}$ and $\rho \in [0, 1]$, which is guaranteed by (P5).

9.1.7 Existence Result

Under the assumptions (P0)—(P5), the twisted primary G-equivariant degree $G\text{-Deg}\,(\mathfrak{F}_\rho, \Omega_{\lambda_1,\lambda_2})$ is well defined and does not depend on the homotopy parameter $\rho \in [0, 1]$.

Definition 9.1.1. We introduce the following notation

$$回 := G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}),$$

we will call 回 the *G-equivariant topological invariant** for the system (9.2).

We have the following result

Theorem 9.1.2. *Under the assumptions (P1)—(P5), if the G-equivariant topological invariant*

$$回 = \sum_{(H)} n_H(H)$$

is nonzero, i.e. there exist a coefficient $n_H \neq 0$ *with* $H = K^{\varphi,l}$, *then there exists* $(\lambda, x) \in \Omega_{\lambda_1,\lambda_2}$ *such that* $\mathfrak{F}_1(\lambda, x) = 0$ *with* $G_x \supset H$. *In other words, there exists a nonconstant* 2π*-periodic solution to (9.2) for some* $\lambda \in [\lambda_1, \lambda_2]$, *and consequently, there is a p-periodic solution to (9.1) with* $p = 2\pi\lambda$. *In addition, if* $H = K^{\varphi,l}$ *is such that* K^φ *is a dominating type in* $\mathbb{H}$, *then there exists a nontrivial periodic solution* $x = x(t)$ *to (9.1) (and consequently a whole G-orbit of solutions) with the exact symmetries* K^φ.

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* We use here the Chinese symbol 回 (*huí*), which means '*return*', i.e. it returns the topological information about the solution set.

9.1.8 Computations of the Equivariant Topological Invariant

Since $\mathfrak{F}_\rho$ is a G-equivariant $\varOmega$-admissible homotopy, we have that

$$\boxdot = G\text{-Deg}\,(\mathfrak{F}_1, \varOmega_{\lambda_1,\lambda_2}) = G\text{-Deg}\,(\mathfrak{F}_0, \varOmega_{\lambda_1,\lambda_2}),$$

where $\mathfrak{F}_0$ is a linearized map given by

$$\mathfrak{F}_0(\lambda, x) := x - \beta(x)\lambda(L + K)^{-1}\,[N_{\mathscr{A}}(\lambda, j(x)) + Kx],$$

on $\varOmega_{\lambda_1,\lambda_2}$. To compute $G\text{-Deg}\,(\mathfrak{F}_0, \varOmega_{\lambda_1,\lambda_2})$, we apply a series of reduction through isotypical decompositions and homotopy deformations. For simplicity, assume $\varGamma$ is a finite group.

Isotypical Decomposition and Related Transformations

Consider the S^1-isotypical decomposition of the space $\mathbb{H}$

$$\mathbb{H} = \mathbb{H}^{S^1} \oplus \mathbb{H}^*$$

where $\mathbb{H}^{S^1} \simeq V$ is composed of constant V-valued functions and $\mathbb{H}^*$ is the orthogonal complement of $\mathbb{H}^{S^1}$.

Put $\varOmega^*_{\lambda_1,\lambda_2} = \varOmega_{\lambda_1,\lambda_2} \cap \big((\lambda_1, \lambda_2) \times \mathbb{H}^*\big)$. For $\lambda \in [\lambda_1, \lambda_2]$, define

$$\mathfrak{F}^*{}_0(\lambda, \cdot) := \mathfrak{F}_0(\lambda, \cdot)|_{\mathbb{H}^*}.$$

Recall that $\mathscr{B} = \mathscr{A}|_V$ (cf. (A1)). For $(\lambda, x) \in (\lambda_1, \lambda_2) \times V$, we have

$$\mathfrak{F}_0(\lambda, x) \equiv -\beta(x)\mathscr{B}(x).$$

Taking into account $\beta(x) \in [\alpha_1, 1]$ $(\alpha_1 > 0)$, we have that $\mathfrak{F}_0|_{\mathbb{H}^{S^1}}$ is G-homotopic to $-\mathscr{B}$. Therefore, the map $\mathfrak{F}_0$ can be viewed as a product map $-\mathscr{B} \times \mathfrak{F}^*_0$ on $B_1(\mathbb{H}^{S^1}) \times \varOmega^*_{\lambda_1,\lambda_2}$. By multiplicativity property of twisted primary degree (cf. Proposition 4.2.6), we obtain

$$G\text{-Deg}\,(\mathfrak{F}_0, \varOmega_{\lambda_1,\lambda_2}) = \varGamma\text{-Deg}(-\mathscr{B}, B_1(\mathbb{H}^{S^1})) \cdot G\text{-Deg}\,(\mathfrak{F}^*_0, \varOmega^*_{\lambda_1,\lambda_2}).$$

Moreover, $\varGamma\text{-Deg}(-\mathscr{B}, B_1(\mathbb{H}^{S^1}))$ can be evaluated by (cf. Subsection 4.1.3)

$$\varGamma\text{-Deg}(-\mathscr{B}, B_1(\mathbb{H}^{S^1})) = \prod_{\mu \in \sigma_-(-\mathscr{B})} \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{m_i(\mu)}, \tag{9.12}$$

where $m_i(\mu)$ is the $\mathcal{V}_i$-multiplicity of μ (cf. (4.4)) and $\deg_{\mathcal{V}_i}$ is the basic degree without parameters associated with $\mathcal{V}_i$ (cf. Definition 4.1.5).

In order to compute $G\text{-Deg}\,(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2})$, we make convenient modifications of the involved maps under admissible homotopies and the sets using excision property. We can assume that

$$\Omega^*_{\lambda_1,\lambda_2} = (\lambda_1, \lambda_2) \times \big(B_2(\mathbb{H}^*) \setminus B_{\frac{1}{2}}(\mathbb{H}^*)\big)$$

and the function β is given by

$$\beta(x) = \begin{cases} 1 & \text{if } \|x\|_{H^1} \leq 1, \\ 2 - \alpha_1 - (1-\alpha_1)\|x\|_{H^1} & \text{if } 1 < \|x\|_{H^1} < 2, \\ \alpha_1 & \text{if } \|x\|_{H^1} \geq 2. \end{cases} \tag{9.13}$$

Consider the further isotypical decomposition

$$\mathbb{H}^* = \overline{\bigoplus_{l=1}^{\infty} \mathbb{H}_l}, \tag{9.14}$$

where each $\mathbb{H}_l$ consists of the functions of form $e^{ilt}z$, $z \in V^c$ (cf. (6.32)). Since $\mathfrak{F}_0^*(\lambda, \cdot)$ is S^1-equivariant, we have $\mathfrak{F}_0^*(\lambda, \cdot)(\mathbb{H}_l) \subset \mathbb{H}_l$ for each $l > 0$. For $\lambda \in [\lambda_1, \lambda_2]$, define $\mathcal{A}_l(\lambda) : \mathbb{H}_l \to \mathbb{H}_l$ by

$$\mathcal{A}_l(\lambda) := \mathfrak{F}_0(\lambda, \cdot)|_{\mathbb{H}_l}.$$

Let $x(t) = e^{ilt}z$ for $z \in V^c$, then

$$\begin{aligned} \mathcal{A}_l(\lambda)(e^{ilt}z) &= e^{ilt}z - \beta(z)\lambda L^{-1}\mathfrak{A}(e^{il(t+\frac{\theta}{\lambda})}z) \\ &= e^{ilt}\left(z - \frac{\beta(z)\lambda}{il}\mathfrak{A}(e^{\frac{il\theta}{\lambda}}z)\right). \end{aligned} \tag{9.15}$$

Based on a similar argument of the splitting lemma (cf. Lemma 3.3.4), we have

$$G\text{-Deg}\,(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2}) = \sum_{l>0} G\text{-Deg}\,(\mathcal{A}_l, \Omega^*_{\lambda_1,\lambda_2} \cap \mathbb{H}_l).$$

Using the identification $\mathbb{H}_l \simeq V^c$, define the linear operator $\mathfrak{A}_l(\lambda, \cdot) : V^c \to V^c$ by

$$\mathfrak{A}_l(\lambda, z) := \mathfrak{A}(e^{\frac{il\theta}{\lambda}}z), \quad z \in V^c.$$

To simplify the computations, we assume that

(B1) For each $l > 0$, the operator $\mathscr{A}_l(\lambda)$ is completely diagonalizable. Every eigenvalue $\mu_{l,k}(\lambda) \in \sigma(\mathscr{A}_l(\lambda))$, for $k = 1, \dots, k_o$, the corresponding eigenspace $\widetilde{E}(\mu_{l,k}(\lambda))$ does not depend on $\lambda \in [\lambda_1, \lambda_2]$.

Denote by $\widetilde{E}_{l,k} := \widetilde{E}(\mu_{l,k}(\lambda))$. Then, $\mathbb{H}_l$ allows a G-isotypical decomposition

$$\mathbb{H}_l = \bigoplus_k \widetilde{E}_{l,k},$$

and we can write

$$\mathscr{A}_l(\lambda) = \bigoplus_k \mu_{l,k}(\lambda)\mathrm{Id}\,.$$

Put

$$\mathcal{A}_{l,k}(\lambda, z) := z - \frac{\beta(z)}{il}\mu_{l,k}(\lambda)z, \quad z \in \widetilde{E}_{l,k}, \tag{9.16}$$

and define the sets

$$\mathcal{U}_{l,k} := \{z \in \widetilde{E}_{l,k} : \frac{1}{2} < \|z\| < 2\}, \quad \Omega_{l,k} := (\lambda_1, \lambda_2) \times \mathcal{U}_{l,k}.$$

Based on a splitting lemma argument (cf. Lemma 3.3.4), we have

$$G\text{-Deg}\,(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2}) = \sum_{l>0}\sum_{k=1}^{k_o} G\text{-Deg}\,(\mathcal{A}_{l,k}, \Omega_{l,k}). \tag{9.17}$$

Reduction to Basic Maps

To compute G-Deg $(\mathcal{A}_{l,k}, \Omega_{l,k})$, introduce the function

$$\varphi_{l,k}(\lambda, t) := 1 - \frac{2 - \alpha_1 - (1 - \alpha_1)t}{il}\mu_{l,k}(\lambda),$$

Then, $\mathcal{A}_{l,k}$ can be rewritten as

$$\mathcal{A}_{l,k}(\lambda, z) = \varphi_{l,k}(\lambda, \|z\|)z, \quad z \in \widetilde{E}_{l,k}. \tag{9.18}$$

Using homotopy property of the twisted primary degree, we may assume that the functions $\varphi_{l,r} : (\lambda_1, \lambda_2) \times (\frac{1}{2}, 2) \to \mathbb{C}$ are continuously differentiable and the sets $\varphi^{-1}_{l,r}(0)$ are composed of a finite number of regular points.

We need the following lemma for the computation of G-Deg $(\mathcal{A}_{l,k}, \Omega_{l,k})$.

Lemma 9.1.3. *Let $U \subset \mathbb{R} \times \mathbb{R}_+$ be an open bounded set and $\varphi : \mathbb{R}^2 \to \mathbb{C}$ a continuously differentiable and U-admissible map such that the set $\Lambda := \varphi^{-1}(0) \cap U$ is composed of regular points of φ. Put*

$$T := \max\{|t| : \exists_\lambda \ (\lambda, t) \in \Lambda\} + 1, \quad \tau := \frac{1}{2}\max\{|t| : \exists_\lambda \ (\lambda, t) \in \Lambda\}.$$

Consider a G-representation $V_{j,l}$ modeled on $\mathcal{V}_{j,l}$, $l > 0$, and define the set

$$\Omega := \{(\lambda, v) \in \mathbb{R} \oplus V_{j,l} : (\lambda, |v|) \in U, \ \ \tau < \|v\| < T\},$$

and the G-equivariant map $\mathcal{A} : \mathbb{R} \oplus V_{j,l} \to V_{j,l}$ by

$$\mathcal{A}(\lambda, v) = \varphi(\lambda, \|v\|) \cdot v.$$

Then $\mathcal{A}$ is Ω-admissible G-equivariant map and

$$G\text{-Deg}\,(\mathcal{A}, \Omega) = \sum_{(\lambda,t)\in\Lambda} \text{sign det}\, D\varphi(\lambda, t) \deg_{\mathcal{V}_{j,l}}.$$

Proof: For every point $(\lambda_o, t_o) \in \Lambda$ we define a small neighborhood Ω_o of the zero set $\{(\lambda_o, v) : \|v\| = t_o\}$ in the space $\mathbb{R} \oplus \mathcal{V}_{j,l}$ by

$$\Omega_i := \{(\lambda, v) : |\lambda - \lambda_o| < \varepsilon_i, \ 0 < t_o - \delta < \|v\| < t_o + \delta\},$$

where δ is chosen to be sufficiently small. Then

$$G\text{-Deg}\,(\mathcal{A}, \Omega) = \sum_{(\lambda_o,t_o)\in\Lambda} G\text{-Deg}\,(\mathcal{A}, \Omega_o),$$

and since for every (λ_o, t_o), the map $\mathcal{A}$ can be approximated on Ω_o by $(\lambda, v) \mapsto D\varphi(\lambda_o, t_o)(\lambda - \lambda_o, \|v\| - t_o)^T \cdot z$, which is clearly homotopic to

$$(\lambda, v) \mapsto J_{i,\pm}(\lambda - \lambda_o, \|v\| - t_o)^T \cdot v,$$

where

$$J_{i,+} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{if sign det}\, D\varphi(\lambda_i, t_i) = 1,$$

$$J_{i,-} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \text{if sign det}\, D\varphi(\lambda_i, t_i) = -1,$$

so the result follows. □

Combining Lemma 9.1.3 with (9.12) and (9.17), we obtain the following computational formula for $\boxdot$.

Theorem 9.1.4. *Under the above assumptions, we have*

$$\boxdot = \prod_{\mu\in\sigma_-(-\mathscr{B})}\prod_{i=0}^{r}\Big(\deg_{\mathcal{V}_i}\Big)^{m_i(\mu)}\cdot$$
$$\sum_{l>0}\sum_{k=1}^{k_o}\sum_{j=0}^{s}\sum_{(\lambda,t)\in\Lambda_{l,k}} m_j(\mu_{l,k}(\lambda))\text{sign}\det D\varphi_{l,k}(\lambda,t)\cdot\deg_{\mathcal{V}_{j,l}}, \tag{9.19}$$

where $m_j(\mu_{l,k}(\lambda)) = \dim(\widetilde{E}_{l,k}\cap U_j)/\dim\mathcal{U}_j$ *is the* $\mathcal{U}_j$*-multiplicity of* $\mu_{l,k}(\lambda)$.

9.2 Symmetric Lotka-Volterra Systems

Throughout this section, we assume that Γ is a finite group and $V := \mathbb{R}^n$ is an orthogonal Γ-representation such that Γ acts on V by permuting the coordinates of vectors $x\in V$.

Consider the following Γ-symmetric Lotka-Volterra type system

$$\dot u(t) = u(t)\cdot\big(r - Au(t-\tau)\big), \tag{9.20}$$

where $u:\mathbb{R}\to V$, $\tau>0$, $r=[r_1,\dots,r_n]^T$, A is an $n\times n$-matrix and '$\cdot$' is the component-wise multiplication, i.e. $u\cdot v = [u_1v_1,\dots,u_nv_n]^T$, for $u = [u_1,\dots,u_n]^T, v=[v_1,\dots,v_n]^T\in V$.

By an appropriate transformation, the problem (9.20) is equivalent to

$$\dot u(t) = -Au(t-\tau)\cdot\big(b+u(t)\big), \tag{9.21}$$

where $b = A^{-1}r$. Let p be the unknown period of a solution u to (9.21). By a change of variable, letting $\lambda=\frac{p}{2\pi}$, $x(t)=u(\lambda t)$, we have that x is a 2π-periodic solution of the problem

$$\dot x(t) = -\lambda Ax(t-\frac{\tau}{\lambda})\cdot\big(b+x(t)\big). \tag{9.22}$$

In what follows we assume that the following conditions hold:

(H0) A and b have positive entries, i.e. $a_{i,j}, b_i > 0$, for $1\le i,j\le n$.

(H1) A is symmetric, positive definite (i.e. $A = A^T$ and $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$) and A is Γ-equivariant. In particular, the matrix

$$B := \operatorname{diag}(b)A,$$

(i.e. $Bx = Ax \cdot b$), where $\operatorname{diag}(b)$ denotes the diagonal matrix $[d_{ij}]$ with $d_{jj} = b_j$, $j = 1, \dots, n$, has only real positive eigenvalues $\mu_1, \cdots, \mu_n$ (not necessarily distinct).

(H2) The vector $b = [b_1, \dots, b_n]^T \in V$ is Γ-invariant, i.e. $\gamma b = b$ for all $\gamma \in \Gamma$.

We make also the following assumption

(H3) For every $\mu \in \sigma(B)$

$$\mu\tau \neq 2n\pi + \frac{\pi}{2}, \quad \text{for all } n \in \mathbb{Z}. \tag{9.23}$$

We are interested in finding a nonstationary periodic solutions of (9.20), which is equivalent to finding a nontrivial 2π-periodic solution of (9.22) for some $\lambda > 0$.

Define $\mathscr{A}$, $\mathscr{R} : C_{V,\tau} \to V$ by

$$\mathscr{A}(u_t) := -Au(t-\tau) \cdot b = -Bu(t-\tau), \tag{9.24}$$

$$\mathscr{R}(u_t) := -Au(t-\tau) \cdot u(0), \tag{9.25}$$

where $u \in C_{V,\tau}$. Notice that, under the assumption (H1), $\mathscr{A}$ and $\mathscr{R}$ satisfy (A1)—(A2). Also, the equation (9.20) is Γ-symmetric by (H0)–(H2). Therefore, we are in the setting discussed in Section 9.1.

9.2.1 Reformulation in Functional Spaces

Following the functional setting presented in Subsection 9.1.3, we take $\mathbb{H}$ defined by (9.3) and the operators L, j, K, $N_{\mathscr{A}}$ and $N_{\mathscr{R}}$ given by (6.16)—(6.18) and (9.5)—(9.6) respectively.

Consider the parameterized systems

$$\begin{cases} \dot{x}(t) = -\alpha\lambda A x(t - \frac{\tau}{\lambda}) \cdot \big(b + x(t)\big), \\ x(0) = x(2\pi), \end{cases} \tag{9.22_α}$$

and

$$\begin{cases} \dot{x}(t) = -\alpha\lambda A x(t - \frac{\tau}{\lambda}) \cdot \big(b + \rho x(t)\big), \\ x(0) = x(2\pi), \end{cases} \tag{9.22$_{\alpha\rho}$}$$

where $\alpha \in (0, 1]$ and $\rho \in [0, 1]$.

Then, (9.22$_\rho$) is equivalent to (cf. (9.7))

$$x - \alpha\lambda(L + K)^{-1}[N_{\mathfrak{A}}(\lambda, j(x)) + \rho N_{\mathfrak{R}}(\lambda, j(x)) + Kx] = 0, \quad x \in \mathbb{H},$$

where $\mathfrak{A}$ and $\mathfrak{R}$ are given by (9.24)—(9.25).

9.2.2 Establishing *A Priori* Bounds

Define a partial order in $V = \mathbb{R}^n$ by

$$x \succ y \iff x_i > y_i, \text{ for all } 1 \le i \le n,$$

where $x = [x_1, \dots, x_n]^T$ and $y = [y_1, \dots, y_n]^T$ are two vectors from $\mathbb{R}^n$. Introduce the following set

$$\mathcal{C} = \{x \in \mathbb{H} : -b \prec x(t) \quad \text{for all} \quad t \in [0, 2\pi]\}.$$

We show that $\mathcal{C}$ verifies the property (P0) in Subsection 9.1.4.

Lemma 9.2.1. *For λ, $\alpha > 0$, every periodic solution $x \in \mathcal{C}$ of (9.22$_\alpha$) satisfies*

$$\int_0^{2\pi} x(t)dt = 0. \tag{9.26}$$

In particular, the equation (9.22$_\alpha$) has no nonzero constant solutions.

Proof: Let $x \in \mathcal{C}$ be a solution to (9.22$_\alpha$), $x(t) = [x_1(t), \dots, x_n(t)]^T$. Then for $k = 1, 2, \dots, n$

$$\dot{x}_k(t) = -\alpha\lambda \sum_j a_{kj} x_j(t - \tau/\lambda) \cdot (b_k + x_k(t)), \tag{9.27}$$

which leads to

$$\frac{\dot{x}_k(t)}{b_k + x_k(t)} = -\alpha\lambda \sum_j a_{kj} x_j(t - \tau/\lambda). \tag{9.28}$$

By integrating (9.28) from 0 to 2π, we obtain (by periodicity of $x(t)$) that

$$\sum_j a_{kj} \int_0^{2\pi} x_j(t-\tau/\lambda)dt = \sum_j a_{kj} \int_0^{2\pi} x_j(t)dt = 0, \quad k = 1, 2, \dots, n.$$

Since the matrix A is invertible, one can easily deduce (9.26). $\square$

The following lemma provides a basis establishing (P2) and also indicates a positive number $\alpha_o \in (0,1)$ satisfying (P1).

Lemma 9.2.2. *(i) For $\lambda_1, \lambda_2 \in \mathbb{R}^1$ with $\lambda_1 < \lambda_2$, there exist a positive number R, and positive Γ-invariant vectors d_1, $d_2 \succ 0$ such that for each $\lambda \in [\lambda_1, \lambda_2]$, $\alpha \in (0,1]$, $\tau \geq 1$, each solution $x \in \mathcal{C}$ of the problem (9.22$_\alpha$) satisfies $\|x\|_{H^1} < R$ and*

$$-b \prec -d_1 \prec x(t) \prec d_2, \quad t \in [0, 2\pi].$$

In addition, there exists $m_o > 0$ such that $\|\dot{x}\|_\infty < m_0$ and $\|\ddot{x}\|_\infty < m_0$.

(ii) There exists $\alpha_o \in (0,1)$ such that there is no nontrivial solution in $\mathcal{C}$ to (9.22$_{\alpha\rho}$) for $\alpha \in (0, \alpha_o]$, $\rho \in [0,1]$ and $\lambda \in [\lambda_1, \lambda_2]$.

Proof: (i) Let $x \in \mathcal{C}$ be a solution to (9.22$_\alpha$), $x(t) = [x_1(t), \dots, x_n(t)]^T$. Then for $k = 1, 2, \dots, n$ we have the relations (9.27) and (9.28) which lead to

$$\ln(b_k + x_k(t)) - \ln(b_k + x_k(s)) = -\alpha\lambda \int_s^t \sum_j a_{kj} x_j(w - \tau/\lambda)dw,$$

where we assume $s \leq t$. Consequently, if s is such that $x_k(s) = 0$ then

$$b_k + x_k(t) = b_k \exp\left(-\alpha\lambda_2 \int_s^t \sum_j a_{kj} x_j(w - \tau/\lambda)dw\right), \quad \text{for all } t \in \mathbb{R}.$$

By the assumptions (H0) and (H1),

$$x_k(t) < d_2^k := b_k \exp\left(2\pi\alpha\lambda_2 \sum_j a_{kj} b_j\right) - b_k \quad \text{for all } t \in \mathbb{R}, \tag{9.29}$$

and (by (H2)) the vector $d_2 := [d_2^1, \dots, d_2^m]^T$ is Γ-invariant. On the other hand,

$$-b_k < -d_1^k := b_k \exp\left(-2\pi\alpha\lambda_1 \sum_j a_{kj} d_2^j\right) - b_k < x_k(t), \quad \text{for all } t \in \mathbb{R},$$

and (by (H2)) the vector $d_1 := [d_1^1, \dots, d_1^n]^T$ is Γ-invariant. By differentiating (9.22_α) we obtain

$$\ddot{x}(t) = -\alpha\lambda\left(A\dot{x}(t-\tau/\lambda)\cdot\big(b + x(t)\big) + Ax(t-\tau/\lambda)\cdot\dot{x}(t)\right). \tag{9.30}$$

By using the above obtained upper and lower bounds for $x_k(t)$ in (9.27) and (9.30), it is easy to show that there exists $m_o > 0$ such that

$$|\dot{x}_k(t)| < m_o \quad \text{and} \quad |\ddot{x}_k(t)| < m_o,$$

for all $k = 1, \dots, n$ and $t \in \mathbb{R}$. Consequently,

$$\|\dot{x}\|_\infty < m_o \text{ and } \quad \|\ddot{x}\|_\infty < m_o.$$

Therefore,

$$\|x\|_{H^1}^2 = \int_0^{2\pi} \dot{x}(t)\dot{x}(t)dt + \int_0^{2\pi} x(t)x(t)dt \le 2\pi\|\dot{x}\|_\infty^2 + 2\pi\sum_{k=1}^{n} d_2^k =: R^2.$$

ii) Suppose for contradiction that there exist sequences $\{\alpha_n\} \subset (0, \alpha_o]$ and $\{x^m\} \in \mathcal{C}$ such that x^m is a non-trivial solution to (9.22_α) for $\alpha = \alpha_m$, $\lambda = \lambda_m \in [\lambda_1, \lambda_2]$ and $\lim_{m\to\infty}\alpha_m = 0$. Then (9.29) holds for $x_k(t) = x_k^m(k)$ with $m = 1, 2, \dots$, and therefore,

$$\lim_{m\to\infty}\|x^m\|_\infty = 0.$$

Since

$$\dot{x}^m(t) = -\alpha_m\lambda Ax^m(t-\tau/\lambda_m)\cdot\big(b + \rho x^m(t)\big), \tag{9.31}$$

we have

$$\|\dot{x}^m\|_\infty \le \alpha_m\lambda_2|A|\|x^m\|_\infty(|b|_\infty + \rho|d_2|_\infty), \tag{9.32}$$

where $|A| = \sum_{ij} a_{ij}$ and $|y| = \max\{|y_j| : j = 1, \dots, n\}$ for $y \in \mathbb{R}^n$. Define $u^m(t)$ by

$$u_k^m(t) = \frac{x_k^m(t)}{\|x^m\|_\infty}, \quad t \in \mathbb{R}.$$

Clearly, $u^m \in \mathbb{H}$ and by (9.32),

$$\|\dot{u}^m\|_\infty \le \alpha_m\lambda_2|A|(|b|_\infty + \rho|d_2|_\infty),$$

which implies that $\lim_{m\to\infty}\|\dot{u}^m\|_\infty = 0$. Since

$$\|u^m\|_\infty \le 2\pi \|\dot{u}^m\|_\infty,$$

it follows that $\lim_{m\to\infty} \|u^m\|_\infty = 0$, which is a contradiction with $\|u^m\|_\infty = 1$. □

We show that $(9.22_{\alpha\rho})$ satisfies (P4) for $\alpha = 1$, $\rho = 0$.

Lemma 9.2.3. *(i) Assume that for a fixed values* $\lambda \in \mathbb{R}^+$ *and* $\alpha \in (0, 1]$, *the linearized equation*

$$\dot{x}(t) = -\alpha\lambda A x(t - \frac{\tau}{\lambda}) \cdot b \tag{9.33}$$

has a nontrivial solution in $\mathbb{H}$. *Then, there exist* k, $n \in \mathbb{Z}$, $n \ge 0$, $k > 0$ *such that*

$$\begin{cases} \lambda = \frac{k\tau}{2\pi n + \pi/2} =: \lambda_{k,n}, \\ \alpha = \frac{k}{\lambda\mu}, \end{cases} \tag{9.34}$$

where μ *is an eigenvalue of the matrix* $B := \mathrm{diag}(b)A$.

(ii) For $\alpha = 1$, $\rho = 0$, *the equation* $(9.22_{\alpha\rho})$ *has no nontrivial solution in* $\mathbb{H}$.

Proof: (i) The equation (9.34) can be written as

$$\dot{x}(t) = -\alpha\lambda B x(t - \tau/\lambda). \tag{9.35}$$

Clearly, (9.35) allows a nontrivial solution u in $\mathbb{H}$ if and only if, there is $k \in \mathbb{N}$ such that $x = e^{ikt} \cdot z$, for some $z \in V^c$, is a solution to (9.35), which leads to the equation

$$ik + \alpha\lambda\mu e^{-ik\frac{\tau}{\lambda}} = 0,$$

for some $\mu \in \sigma(B)$. One can easily verify that such a case is possible if and only if, the relations (9.34) are satisfied for some $n \in \mathbb{Z}$.

(ii) If $\alpha = 1$, then $(9.22_{\alpha\rho})$ reduces to (9.34). By (i), a nontrivial solution to (9.34) implies that $\mu\tau = 2\pi n + \pi/2$, which contradicts the assumption (H3). □

The lemma below provides a positive number m_1 satisfying (P3).

Lemma 9.2.4. *Assume that* $\lambda \in \mathbb{R}^+$, $\rho \in [0, 1]$ *and* $\alpha \in (0, 1]$ *are fixed.*

(i) If zero is not an isolated solution in $\mathbb{H}$ *to the equation* $(9.22_{\alpha\rho})$, *then there exist integers* $k > 0$ *and* $n \le 0$ *such that* λ *and* α *satisfy the relations (9.34) for an eigenvalue* μ *of the matrix* B.

(ii) If λ_1*,* $\lambda_2 \in \mathbb{R}^+$ *with* $\lambda_1 < \lambda_2$*, then there exists* $m_1 > 0$ *such that for all* $\lambda \in [\lambda_1, \lambda_2]$*, the equation (9.22*$_\rho$*) has no non-trivial solution* $x \in \mathbb{H}$ *such that* $\|x\|_{H^1} \leq m_1$*.*

Proof: (i) Define $\mathfrak{F}_\alpha : [\lambda_1, \lambda_2] \times \mathbb{H} \to \mathbb{H}$ by

$$\mathfrak{F}_\alpha(\rho, \lambda, x) := x - \alpha\lambda(L + K)^{-1}\mathcal{N}(\rho, \lambda, x), \quad x \in \mathbb{H}.$$

By implicit function theorem, if $(\lambda, 0)$ is not an isolated solution to $(9.22_{\alpha\rho})$ for some $\rho \in [0, 1]$, then $D_x\mathfrak{F}_\alpha(\rho, \lambda, 0) : \mathbb{H} \to \mathbb{H}$ is not an isomorphism, which implies that the equation (9.33) has a nontrivial solution. Consequently, by Lemma 9.2.3, α and λ satisfy the relations (9.34).

(ii) The equation (9.22_ρ) is a special case of $(9.22_{\alpha\rho})$ for $\alpha = 1$. Assume such $m_1 > 0$ does not exist, then zero is not an isolated solution in $\mathbb{H}$. By (i), then the relations (9.34) have solutions for an eigenvalue $\mu \in \sigma(B)$. Since $\alpha = 1$, we have $\mu\tau = 2\pi n + \pi/2$, which contradicts the assumption (H3). ☐

The following fact shows that (P5) can be achieved for specific choices of λ_1, λ_2 (cf. (9.38)). For the sake of completeness, we include its elementary proof.

Lemma 9.2.5. *For any* $\rho \in [0, 1]$ *and* $\lambda > 0$*, the following equation*

$$\begin{cases} \dot{x}(t) = -\lambda Ax(t) \cdot \big(b + \rho x(t)\big), \\ x_0 = x_{2\pi}. \end{cases} \tag{9.36}$$

has no nontrivial solution.

Proof: Assume first that $\rho \in (0, 1]$. Suppose that x is a non-zero 2π-periodic solution to (9.36). By integrating (9.2.5) from 0 to 2π, we obtain

$$\int_0^{2\pi} Ax(t) \cdot x(t)dt = 0 \iff \sum_{j=1}^{n} a_{kj} \int_0^{2\pi} x_j(t)x_k(t)dt = 0,\ k = 1, 2, \ldots, n. \tag{9.37}$$

On the other hand, A is positively definite, i.e. $Ax(t) \bullet x(t) > 0$ for $x(t) \neq 0$, which implies that

$$\int_0^{2\pi} Ax(t) \bullet x(t)dt > 0.$$

But this is a contradiction, because by summing up the equations in (9.37), we obtain

$$\int_0^{2\pi} Ax(t) \bullet x(t)dt = \sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj}\int_0^{2\pi} x_j(t)x_k(t)dt = 0.$$

Suppose now that $\rho = 0$, then the equation (9.36) becomes $\dot{x}(t) = -\lambda Bx(t)$. Consequently, if x is a 2π-periodic solution to (9.36) for $\rho = 0$, then it also satisfies the equation

$$\frac{d}{dt}(x(t)\cdot x(t)) = 2\dot{x}(t)\cdot x(t) = -2\lambda Bx(t)\cdot x(t),$$

which leads to

$$\int_0^{2\pi} Bx(t)\cdot x(t)dt = 0.$$

Be a similar argument as above, we obtain again that $x(t) = 0$. □

Therefore, by Lemmas 9.2.1–9.2.5, we established the *a priori* bounds for (9.22_α) and ($9.22_{\alpha\rho}$) which satisfy properties (P0)—(P1), (P3)—(P4).

9.2.3 Sets and Deformations

For fixed $\lambda_1, \lambda_2 \in \mathbb{R}^1$ with $\lambda_1 < \lambda_2$ and assume $d_2 \succ \frac{b_1+d_1}{2}$. We define the following $\Gamma \times S^1$-invariant sets

$$\begin{aligned}
\mathcal{D} &:= \{x \in \mathbb{H} : -\frac{b+d_1}{2} \prec x(t) \prec 2d_2,\ t \in [0,2\pi]\},\\
\widetilde{\mathcal{D}} &:= \{x \in \mathbb{H} : -d_1 \prec x(t) \prec d_2,\ t \in [0,2\pi]\},\\
\overline{\mathcal{B}} &:= \{x \in \mathbb{H} : \|x\|_{H^1} \le m_1\},\\
B_R &:= \{x \in \mathbb{H} : \|x\|_{H^1} < R\},
\end{aligned}$$

where R, d_1 and d_2 are specified in Lemma 9.2.2 and m_1 in Lemma 9.2.4. We can choose $m_1 > 0$ to be sufficiently small so that

$$\overline{\mathcal{B}} \subsetneq \widetilde{\mathcal{D}} \subsetneq \overline{\mathcal{D}} \subsetneq \mathcal{C}.$$

and define

$$\widetilde{\mathcal{U}} := (\widetilde{\mathcal{D}} \cap B_R).$$

Choose $\varepsilon > 0$ to sufficiently small such that the set

$$\mathcal{U} := \{x \in \mathbb{H} : \operatorname{dist}(x, \widetilde{\mathcal{U}}) < \varepsilon\},$$

satisfies $\overline{\mathcal{U}} \subset \mathcal{D}$. Thus, the sets $\widetilde{\mathcal{U}}, \mathcal{U}$ satisfy (P2) by Lemma 9.2.2(i).

Next, we choose λ_1 and λ_2 to be

$$\lambda_1 := \frac{\tau}{2j_1\pi}, \quad \lambda_2 := \frac{\tau}{2j_2\pi}, \quad j_1 > j_2,\ j_1, j_2 \in \mathbb{N}. \tag{9.38}$$

Then, by Lemma 9.2.5, λ_1, λ_2 satisfy (P5). Define the set $\Omega_{\lambda_1,\lambda_2} \subset \mathbb{R}^+ \times \mathbb{H}$ by (9.8).

Based on the above discussion, we established the *a priori* bounds for (9.22_α) and $(9.22_{\alpha\rho})$ which verify (P0)—(P5). Thus, $\mathfrak{F}_\rho$ defined by (9.11) is indeed an $\Omega_{\lambda_1,\lambda_2}$-admissible homotopy. Therefore, the equivariant topological invariant $\boxed{\square}$ is well-defined (cf. Definition 9.1.1) and the computational formula (9.19) is valid.

9.2.4 Computation of the Equivariant Topological Invariant

To determine the negative spectrum of $-\mathscr{B}$, observe that $\mathscr{B} = \mathscr{A}|_V = -B$ (cf. (9.24)). By (H1), the matrix B has only positive eigenvalues. Thus,

$$\sigma_-(-\mathscr{B}) = \sigma_-(B) = \emptyset.$$

Therefore, the computational formula (9.19) reduces to

$$\begin{aligned}\boxed{\square} &= (\Gamma) \cdot \sum_{l>0} \sum_{k=1}^{k_o} \sum_{j=0}^{s} \sum_{(\lambda,l)\in\Lambda_{l,k}} m_j(\mu_{l,k}(\lambda))\text{sign det } D\varphi_{l,k}(\lambda, l) \cdot \deg_{\mathcal{V}_{j,l}} \\ &= \sum_{l>0} \sum_{k=1}^{k_o} \sum_{j=0}^{s} \sum_{(\lambda,t)\in\Lambda_{l,k}} m_j(\mu_{l,k}(\lambda))\text{sign det } D\varphi_{l,k}(\lambda, l) \cdot \deg_{\mathcal{V}_{j,l}}.\end{aligned} \tag{9.39}$$

By direct computation, we have (cf. (9.15))

$$\mathcal{A}_l(\lambda)(e^{ilt}z) = e^{itl}\left[z + \frac{\beta(z)}{il}e^{-\frac{il\tau}{\lambda}}Bz\right], \quad z \in V^c.$$

Take $\mu_{l,k} \in \sigma(B)$, we write (cf. (9.16))

$$\mathcal{A}_{l,k}(\lambda)(z) = z + \frac{\beta(z)\mu_{l,k}}{il}e^{-\frac{il\tau}{\lambda}}z, \quad z \in \widetilde{E}_{l,k},$$

where β is defined by (9.13). To determine the function $\varphi_{l,k}$ according to (9.18), we express $\mathcal{A}_{l,k}$ as

$$\begin{aligned}\mathcal{A}_{l,k}(\lambda)(z) &= \Big(1+\frac{\beta(z)\mu_{l,k}}{il}e^{-\frac{il\tau}{\lambda}}\Big)z\\ &= \Big(1+\frac{(2-\alpha_1-(1-\alpha_1)\|z\|)\mu_{l,k}}{il}e^{-\frac{il\tau}{\lambda}}\Big)z, \quad z\in\Omega_{\lambda_1,\lambda_2}\cap\widetilde{E}_{l,k}.\end{aligned}$$

Then, $\varphi_{l,k}:\mathbb{R}^2\to\mathbb{C}$ is defined by

$$\varphi_{l,k}(\lambda,t) := 1+\frac{(2-\alpha_1-(1-\alpha_1)t)\mu_{l,k}}{il}e^{-\frac{il\tau}{\lambda}}.$$

To simplify notations, put $\xi(t) := 2-\alpha_1-(1-\alpha_1)t$ for all $t\in(1,2)$. Notice that $\beta(z)\equiv\xi(\|z\|)$ for $1<\|z\|<2$. To compute $\boxdot$ according to (9.39), we need to differentiate $\varphi_{l,k}$ at the point (λ_o,t_o) satisfying (cf. (9.34)

$$\begin{cases}\xi(t_o)=\frac{l}{\lambda_o\mu_{l,k}}<1,\\ \lambda_o:=\lambda_{l,m}=\frac{l\tau}{2\pi m+\pi/2}, \quad \text{for some } \ m\in\mathbb{N}.\end{cases} \tag{9.40}$$

We have that

$$\begin{aligned}\varphi_{l,k}(\lambda,t) &= 1+\frac{\xi(t)\mu_{l,k}}{il}e^{-\frac{il\tau}{\lambda}}\\ &= 1-\frac{\xi(t)\mu_{l,k}}{l}\sin\frac{l\tau}{\lambda}-i\frac{\xi(t)\mu_{l,k}}{l}\cos\frac{l\tau}{\lambda}.\end{aligned}$$

Then, we obtain

$$D\varphi_{l,k}(\lambda,t)=\begin{bmatrix}\frac{\xi(t)\mu_{l,k}}{l}\cos\frac{l\tau}{\lambda}(-\frac{l\tau}{\lambda^2}) & \frac{(1-\alpha_1)\mu_{l,k}}{l}\sin\frac{l\tau}{\lambda}\\ \frac{\xi(t)\mu_{l,k}}{l}\sin\frac{l\tau}{\lambda}(-\frac{l\tau}{\lambda^2}) & \frac{(1-\alpha_1)\mu_{l,k}}{l}\cos\frac{l\tau}{\lambda}\end{bmatrix},$$

which evaluated at (λ_o,t_o) gives (notice that $\cos\frac{l\tau}{\lambda_o}=0$ and $\sin\frac{l\tau}{\lambda_o}=1$)

$$D\varphi_{l,k}(\lambda_o,t_o)=\begin{bmatrix}0 & \frac{(1-\alpha_1)\mu_{l,k}}{l}\\ -\frac{l\tau}{\lambda_o^3} & 0\end{bmatrix}.$$

Clearly, sign det $D\varphi_{l,k}(\lambda_o,t_o)>0$. Thus (cf. (9.39)),

$$\boxdot=\sum_{l>0}\sum_{k=1}^{k_o}\sum_{j=0}^{s}\sum_{(\lambda,t)\in\Lambda_{l,k}} m_j(\mu_{l,k})\deg_{\mathcal{V}_{j,l}}.$$

Finally, to determine $\Lambda_{l,k}$, for $\mu_{l,k} \in \sigma(B)$, denote by $n(\mu_{l,k})$ a positive integer such that

$$\frac{\pi}{2} + 2n(\mu_{l,k})\pi < \mu_{l,k}\tau < \frac{\pi}{2} + 2(n(\mu_{l,k}) + 1)\pi.$$

Then, we have (cf. (9.40))

$$\begin{aligned}\Lambda_{l,k} :=\{(\lambda_o, l_o) \,:\, \lambda_o = \frac{l\tau}{2\pi m + \pi/2},\ \xi(l_o) = \frac{l}{\lambda_o \mu_{l,k}},\\ lj_2 \leq m < lj_1,\ \ n(\mu_{l,k}) \geq m\}.\end{aligned}$$

Theorem 9.2.6. *Under the assumptions (H0)–(H3), if the G-equivariant topological invariant*

$$\boxdot = \sum_{(H)} n_H(H)$$

is nonzero, i.e. there exist a coefficient $n_H \neq 0$ with $H = K^{\varphi,l}$, then there exists $(\lambda, x) \in \Omega_{\lambda_1,\lambda_2}$ such that $\mathfrak{F}_1(\lambda, x) = 0$ with $G_x \supset H$. In other words, there exists a nonconstant 2π-periodic solution to (9.22) for some $\lambda \in [\lambda_1, \lambda_2]$, and consequently, there is a p-periodic solution to (9.20) with $p = 2\pi\lambda$. In addition, if $H = K^{\varphi,l}$ is a dominating type in $\mathbb{H}$, then there exists a nontrivial periodic solution $x(t)$ to (9.20) (and consequently a whole G-orbit of solutions) with the exact symmetries K^{φ}.

As an immediate consequence, we obtain the following generalization of the result obtained in [90] (without assumption of simplicity on the eigenvalues of the matrix B)

Corollary 9.2.7. *Suppose that $\Gamma = \{e\}$. Under the assumptions (H0)—(H3), if there exist an eigenvalue $\mu \in \sigma(B)$ and $n \in \mathbb{N} \cup \{0\}$ such that*

$$\frac{\pi}{2} + 2n\pi < \mu\tau < \frac{\pi}{2} + 2(n+1)\pi,$$

then the G-equivariant topological invariant

$$\boxdot = \sum_{(H)} n_H(H)$$

is nonzero, and consequently, there exists a p-periodic solution to (9.20).

9.3 Usage of Maple© Routines and Computational Examples

In the computational examples, we consider the system (9.21) symmetric with respect to Γ being Q_8, D_8 and S_4. In addition, we assume that $b = [1, 1, \dots, 1]^T$. For each considered matrix $A = B$, we choose concrete numerical values of its entries, as well we also specify the numerical value of the delay $\tau > 0$. The spectrum of A will be denoted by $\{\mu_k : 1 \le k \le k_o\}$, and the corresponding to μ_k eigenspace $E(\mu_k)$ will turn out to be of a single Γ-isotypical type, i.e. $E(\mu_k) = m_{i(k)}(\mu_k) \cdot \mathcal{V}_{i(k)}$, where $m_{i(k)}(\mu_k)$ denotes the $\mathcal{V}_{i(k)}$-multiplicity of the eigenvalue of μ_k. In all considered cases, we always have $m_i(\mu_k) = 1$. Similarly, the for the matrix $A : V^c \to V^c$ we will denote by $\widetilde{E}(\mu_k)$ the (complex) eigenspace, which in our cases will be $\widetilde{E}(\mu_k) = m_{j(k)}(\mu_k) \cdot \mathcal{U}_{j(k)}$, where $m_{j(k)}(\mu_k)$ is the $\mathcal{U}_{j(k)}$-multiplicity of μ_k. The number $m_{j(k)}(\mu_k)$ will be always one, except for one eigenvalue in the case $\Gamma = Q_8$, where the considered (real) eigenspace will be of quaternionic type, so this number is equal 2.

We choose the values of $j_1 = 1$ and $j_2 = 1$, and put

$$\mathfrak{m}_{l,j} := m_{j(k)}(\mu_k)|\Lambda_{l,k}|, \qquad \text{where } \widetilde{E}(\mu_k) = m_{j(k)}(\mu_k) \cdot \mathcal{U}_{j(k)},$$

and $|X|$ denotes the number of elements in the set X. Then, using this notation, our computational formula for the associated equivariant twisted degree can be simplified as follows

$$\begin{aligned} \boxdot &= \sum_{l>0} \sum_{\mu_{l,k} \in \sigma(B)} \sum_{(\lambda,t) \in \Lambda_{l,k}} \sum_{j=0}^{s} m_j(\mu_{l,k}) \deg_{\mathcal{V}_{j,l}} \\ &= \sum_{l>0} \sum_{j=1}^{s} \mathfrak{m}_{j,l} \deg_{\mathcal{V}_{j,l}}. \end{aligned} \tag{9.41}$$

For the computation of the numbers $n(\mu_i)$, we use Table 9.1. The final

n	1	2	3	4	5	6	7	8	9	10
$\frac{\pi}{2} + 2n\pi$	7.9	14.1	20.4	26.7	33.0	39.27	45.6	51.8	58.1	64.4

Table 9.1. Values of $\frac{\pi}{2} + 2n\pi$.

results are formulated in basic degrees $\deg_{\mathcal{V}_{j,l}}$. For the values of basic degrees

$\deg_{\mathcal{V}_{j,1}}$, we refer to Appendix A2.3. The degrees $\deg_{\mathcal{V}_{j,l}}$ can be determined by taking the l-folding homomorphism of $\deg_{\mathcal{V}_{j,1}}$, i.e. $\deg_{\mathcal{V}_{j,l}} = \Psi_l(\deg_{\mathcal{V}_{j,1}})$, for $\Psi_l : A_1^t(G) \to A_1^t(G)$ defined (on generators) by $(H^{\varphi,k}) \mapsto (H^{\varphi,kl})$.

For each non-zero coefficient in $\boxdot$ of $(H^{\varphi,l})$, where (H^{φ}) is a dominating orbit type, there exist at least $|\Gamma/H|$ different non-constant p-periodic solutions with the least symmetry $(H^{\varphi,k})$ for some integer $k \geq 1$. However, the k-folding in the isotropy group $(H^{\varphi,k})$ of $x \in \mathbb{H}^*$ means that x is a p/k-periodic solution with symmetries exactly (H^{φ}). In this way we are able to predict the exact symmetries of certain periodic solutions.

In Appendix A4.4, we list existence results for the Γ-symmetric Lotka-Volterra type systems, for Γ being the quaternionic group Q_8, the dihedral group D_8 and the octahedral group S_4.

10

Existence of Periodic Solutions to Symmetric Variational Problems

In this chapter, we study the existence of periodic solutions to symmetric variational problems. More precisely, we first investigate the existence of nonstationary periodic solutions to an autonomous Newtonian system of describing trajectories of finitely many particles, governed by the Newton's laws of motion. As sufficient differentiability of the force function is stipulated, the Newtonian system of our consideration is energy conserving, thus all variational techniques apply.

We consider an autonomous Newtonian system *symmetric* with respect to a compact Lie group Γ, which acts on the phase space V. The Γ-equivariant nature of the force function leads to a $\Gamma \times S^1$-equivariant variational problem, where periodic solutions to a Γ-symmetric autonomous Newtonian system correspond naturally to critical points of the associated $\Gamma \times S^1$-invariant total energy functional Ψ.

To the gradient map of the energy functional, which is assumed to be asymptotically linear at ∞, we associate two topological invariants $\deg_0$ and $\deg_\infty$, representing the gradient $\Gamma \times S^1$-degrees of $\nabla\Psi$ on a small ball B_ε and a large ball B_R, respectively. The difference $\deg_\infty - \deg_0$ is the topological invariant capturing the existence of nonstationary periodic solutions to the system in $B_R \setminus B_\varepsilon$.

Then, we study an $O(2)$-symmetric elliptic problem with periodic-Dirichlet mixed boundary conditions. By a similar procedure, we obtain the existence result.

The chapter is organized as follows. In Section 10.1, we discuss a symmetric autonomous Newtonian system having 0 and ∞ as *non-degenerate* critical points of the energy functional. In this case, the standard linearization technique applies. Consequently, the computations of the topological invariants $\deg_p$ ($p \in \{0, \infty\}$) reduce to the computations of gradient linear isomorphisms, which adopt the effective computational formulae discussed in Subsection 5.2.2. The computational examples are provided in Appendix A4.5 for $\Gamma = D_6, S_4, A_5$. In Section 10.2, we extend our discussion to the symmetric autonomous Newtonian system allowing *degenerate* critical points at 0 and/or ∞. Applying a result of splitting lemmas (cf. [69]), we obtain a product type of formula for each $\deg_p$ ($p \in \{0, \infty\}$), which is only computable up to an unknown factor (due to the degeneracy of the system). Under certain assumptions, the invariant $\deg_\infty - \deg_0$ still contains enough information about the symmetric structure of the solution set. Numerical illustrations will be provided in Appendix A4.6 for Γ being dihedral groups D_6, D_8, D_{10} and D_{12}. In Section 10.3, we study an $O(2)$-symmetric asymptotically linear elliptic equation with periodic-Dirichlet mixed boundary conditions. By applying a similar degree-theoretical procedure, we obtain the existence result of at least two different types of periodic solutions. Computational example is provided in Example 10.3.3.

10.1 Symmetric Autonomous Newtonian System

Throughout this section, Γ is a finite group, V is an orthogonal Γ-representation and $\varphi : V \to \mathbb{R}$ is a C^2-differentiable Γ-invariant function. Then, the gradient map $\nabla\varphi : V \to V$ is a C^1-differentiable Γ-equivariant map.

We are interested in finding nonzero solutions to the following Γ-symmetric autonomous Newtonian system

$$\begin{cases} \ddot{x} = -\nabla\varphi(x), & x(t) \in V, \\ x(0) = x(2\pi), & \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{10.1}$$

where $x : \mathbb{R} \to V$ is twice weakly differentiable with respect to t and $\nabla\varphi$ satisfies that

(A1) $\nabla\varphi(x) = 0 \iff x = 0$.

In addition, there exist two symmetric Γ-equivariant linear isomorphisms $A, B : V \to V$ such that

(A2) $\nabla^2\varphi(0) = A$.
(A3) $\nabla\varphi(x) = Bx + o(\|x\|)$ as $\|x\| \to \infty$, i.e.

$$\lim_{\|x\|\to\infty} \frac{\|\nabla\varphi(x) - Bx\|}{\|x\|} = 0.$$

Notice that the conditions (A1)—(A3) imply that

$$\Gamma\text{-Deg}(-A, B_1(V)) = \Gamma\text{-Deg}(-B, B_1(V)). \tag{10.2}$$

Indeed, by the standard linearization argument and (A2), there exists $\varepsilon > 0$ such that

$$\Gamma\text{-Deg}(-A, B(V)) = \Gamma\text{-Deg}(-A, B_\varepsilon(V)) = \Gamma\text{-Deg}(-\nabla\varphi, B_\varepsilon(V)).$$

Similarly, using (A3), for $R > 0$ being sufficiently large number, we have

$$\Gamma\text{-Deg}(-B, B(V)) = \Gamma\text{-Deg}(-B, B_R(V)) = \Gamma\text{-Deg}(-\nabla\varphi, B_R(V)).$$

However, (A1) forces $-\nabla\varphi^{-1}(0) = \{0\}$, by excision property of the Γ-equivariant degree, we have

$$\Gamma\text{-Deg}(-\nabla\varphi, B_\varepsilon(V)) = \Gamma\text{-Deg}(-\nabla\varphi, B_R(V)).$$

Therefore, (10.2) follows.

The following assumption allows the system (10.1) having non-degenerate linearization at 0 and ∞.

(A4) $(\sigma(A) \cup \sigma(B)) \cap \{k^2 : k = 0, 1, 2, \dots\} = \emptyset$,

where $\sigma(A)$ (resp. $\sigma(B)$) denotes the spectrum of A (resp. the spectrum of B).

Remark 10.1.1. Suppose that $C : V \to V$ is a symmetric linear operator such that $\sigma(C) \cap \{k^2 : k = 0, 1, 2, \dots\} = \emptyset$, then the system

$$\begin{cases} -\ddot{x} = Cx, & x(t) \in V, \\ x(0) = x(2\pi), & \dot{x}(0) = \dot{x}(2\pi) \end{cases}$$

has no non-zero solutions. Therefore, the condition (A4) implies that the linearization of (10.1) at $x = 0$ and $x = \infty$ have no non-zero solutions.

Example 10.1.2. One can easily construct an example of a Γ-invariant function $\varphi : V \to \mathbb{R}$ satisfying the assumptions (A1)—(A4). For instance, let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^2-differentiable function such that $\eta'(t) > 0$ for all $t \in \mathbb{R}$ and $\lim_{t\to\infty} \eta'(t) = b > 0$. Also, assume that $2\eta'(0), 2b \notin \{k^2 : k = 0, 1, 2, \dots\}$. Then, $\varphi(x) := \eta(\|x\|^2)$ is Γ-invariant and the gradient $\nabla\varphi(x) = 2\eta'(\|x\|^2)x$, satisfies (A1) and clearly $\nabla\varphi(0)h = 2\eta'(0)h$.

On the other hand,

$$\lim_{\|x\|\to\infty}\frac{\|\nabla\varphi(x)-2bx\|}{\|x\|}=\lim_{\|x\|\to\infty}\frac{\|(2\eta'(\|x\|^2)-2b)x\|}{\|x\|}$$
$$=\lim_{\|x\|\to\infty}|2\eta'(\|x\|^2)-2b|=0,$$

so (A2) and (A3) are clearly satisfied with $A=2\eta'(0)\,\mathrm{Id}$, $B=2b\,\mathrm{Id}$.

10.1.1 Functional Setting

The system (10.1) can be reformulated as a variational problem in the Sobolev space $W:=H^1(S^1;V)$, which is a natural isometric Hilbert G-representation for $G=\Gamma\times S^1$, with the G-action given by (cf. 6.21) and the inner product defined by

$$\langle u,v\rangle_{H^1}:=\int_0^{2\pi}\langle\dot u(t),\dot v(t)\rangle+\langle u(t),v(t)\rangle dt,\quad u,v\in W.$$

We will denote by $\|\cdot\|_{H^1}$ the induced norm by $\langle\cdot,\cdot\rangle_{H^1}$ on W.

Define $\Psi:W\to\mathbb{R}$ by

$$\Psi(u):=\int_0^{2\pi}\left(\frac{1}{2}\|\dot u(t)\|^2-\varphi(u(t))\right)dt,\tag{10.3}$$

(where $\|\cdot\|$ stands for the L^2-norm). Clearly, the functional Ψ is G-invariant and C^2-differentiable. Indeed, one can easily verify that

$$D\Psi(u)(v)=\int_0^{2\pi}\langle\dot u(t),\dot v(t)\rangle-\langle\nabla\varphi(u(t)),v(t)\rangle\,dt.$$

Notice that if $D\Psi(u)\equiv0$ for some $u\in W$, then $u\in H^2(S^1;V)$ and u is a solution to (10.1). Consequently, the problem (10.1) can be reformulated as

$$\nabla\Psi(u)=0.\tag{10.4}$$

To determine an explicit formula for $\nabla\Psi$, we represent Ψ as

$$\Psi(u)=\frac{1}{2}\|u\|_{H^1}^2-\widetilde{\Phi}(u),\quad u\in W,$$

where

$$\widetilde{\Phi}(u) = \int_0^{2\pi} \widetilde{\varphi}(u(t))dt, \quad \widetilde{\varphi}(h) = \varphi(h) + \frac{1}{2}\|h\|^2, \quad h \in V.$$

Clearly, $\nabla\Psi(u) = u - \nabla\widetilde{\Phi}(u)$.

Introduce the following maps:

$$\begin{aligned} L &: H^2(S^1;V) \to L^2(S^1;V), & Lu &= -\ddot{u} + u, \\ j &: H^2(S^1;V) \to H^1(S^1;V), & ju &= u, \\ N_{\nabla\widetilde{\varphi}} &: C(S^1;V) \to L^2(S^1;V), & N_{\nabla\widetilde{\varphi}}(u) &= \nabla\widetilde{\varphi}(u) = \nabla\varphi(u) + \mathrm{Id}. \end{aligned} \tag{10.5}$$

Since the equation

$$\langle \nabla\widetilde{\Phi}(u), v\rangle_{H^1} = D\widetilde{\Phi}(u)(v),$$

translates to

$$\int_0^{2\pi}\left(\left\langle \frac{d}{dt}\nabla\widetilde{\Phi}(u)(t), \dot{v}(t)\right\rangle + \langle \nabla\widetilde{\Phi}(u)(t), v(t)\rangle\right)dt = \int_0^{2\pi}\langle \nabla\widetilde{\varphi}(u(t)), v(t)\rangle\, dt,$$

for all $v \in H^1(S^1;V)$, we obtain that $\nabla\widetilde{\Phi}(u)$ is a weak solution y to the system

$$\begin{cases} -\ddot{y} + y = \nabla\widetilde{\varphi}(u), \\ y(0) = y(2\pi),\ \dot{y}(0) = \dot{y}(2\pi). \end{cases}$$

Therefore, one obtains

$$\nabla\widetilde{\Phi}(u) = j \circ L^{-1} \circ N_{\nabla\widetilde{\varphi}}(u), \quad u \in W,$$

which leads to

$$\nabla\Psi(u) = u - j \circ L^{-1} \circ N_{\nabla\widetilde{\varphi}}(u), \quad u \in W.$$

Therefore,

$$x \quad \text{is a solution to (10.1)} \quad \iff \quad \nabla\Psi(x) = 0, \quad x \in W.$$

Notice that since j is a compact inclusion, the gradient G-map $\nabla\Psi$ is indeed a completely continuous G-equivariant field on W, and the gradient equivariant degree method applies.

By (A2)–(A4), for sufficiently small $\varepsilon > 0$ (resp. sufficiently large $R > 0$), the map $\nabla\Psi$ is $B_\varepsilon(W)$-admissible (resp. $B_R(W)$-admissible). Thus, one can define the following gradient G-equivariant degrees

$$\deg_0 := \nabla_G\text{-deg}\,(\nabla\Psi, B_\varepsilon(W)),$$
$$\deg_\infty := \nabla_G\text{-deg}\,(\nabla\Psi, B_R(W)).$$

By the excision property of the gradient degree, if $\deg_\infty - \deg_0 \neq 0$, then there exists a solution to (10.4) and equivalently, to the system (10.1), in $B_R(W) \setminus B_\varepsilon(W)$ (cf. [69]).

10.1.2 Existence Result

Define the G-orthogonal isomorphisms $\mathcal{A}, \mathcal{B} : W \to W$ by

$$\mathcal{A} := \text{Id} - j \circ L^{-1} \circ (A + \text{Id}), \quad \mathcal{B} := \text{Id} - j \circ L^{-1} \circ (B + \text{Id}). \tag{10.6}$$

By (A2) (A4) and the linearization argument, we have

$$\deg_0 = \nabla_G\text{-deg}\,(\mathcal{A}, B_1(W)), \tag{10.7}$$
$$\deg_\infty = \nabla_G\text{-deg}\,(\mathcal{B}, B_1(W)), \tag{10.8}$$

which leads to the following existence result for the system (10.1).

Theorem 10.1.3. *Let $G = \Gamma \times S^1$ for Γ being finite. Consider a Γ-orthogonal representation V and a Γ-equivariant C^2-differentiable function $\varphi : V \to \mathbb{R}$ satisfying verifying (A1) (A4). Suppose that the maps $\mathcal{A}$ and $\mathcal{B}$ are given by (10.6) with*

$$\deg_\infty - \deg_0 = (\deg^0, -\deg^t) \in A(\Gamma) \oplus A_1^t(G) \simeq U(G). \tag{10.9}$$

Then, $\deg^0 = 0$ *and if*

$$\deg^t = \sum_{(H)} n_H \cdot (H) \neq 0$$

i.e. $n_{H_o} \neq 0$, for some orbit type (H_o) in W, then there exists a non-constant periodic solution x_o to (10.1) satisfying $G_{x_o} \supset H_o$. In addition, if $H_o = K_o^{\psi,k}$ is such that $(K_o^{\psi,1})$ is a dominating orbit type in W, then there exist at least $|\Gamma/K_o|$ different non-constant periodic solutions with the orbit type at least $(K_o^{\psi,k})$.

Proof: By definition of the gradient equivariant degree (cf. (5.24)-(5.25)),

$$\deg^0 = \Gamma\text{-Deg}(\mathcal{B}|_{W^{S^1}}, B_1(W^{S^1})) - \Gamma\text{-Deg}(\mathcal{A}|_{W^{S^1}}, B_1(W^{S^1})).$$

Observe that $W^{S^1} \simeq V$ and $\mathcal{A}|_{W^{S^1}} = -A$, $\mathcal{B}|_{W^{S^1}} = -B$ (cf. (10.6)). Thus,

$$\deg^0 = \Gamma\text{-Deg}(-B, B_1(V)) - \Gamma\text{-Deg}(-A, B_1(V)).$$

Combined with (10.2), we conclude $\deg^0 = 0$.

By (10.7)—(10.8) and the excision property of gradient equivariant degree, if

$$\nabla_G\text{-deg}\,(\mathcal{A}, B(W)) - \nabla_G\text{-deg}\,(\mathcal{B}, B(W)) \neq 0,$$

then there exists a solution to the system (10.1) in $B_R(W) \setminus B_\varepsilon(W)$. Moreover, by (A1), $x = 0$ is the only constant solution to (10.1). Therefore, there exists a non-constant solution to the system (10.1) in $B_R(W) \setminus B_\varepsilon(W)$.

Suppose that $n_{H_o} \neq 0$, where $(H_o) = (K_o^{\psi,k})$ and $(K_o^{\psi,1})$ is a dominating orbit type in W. Then, by the existence property of gradient equivariant degree, there exists a solution $u \in B_R(W) \setminus B_\varepsilon(W)$ to the system (10.1) such that $G_u \supset H_o$. Due to (A1), we have that $(G_u) = (K^{\widetilde{\psi},\widetilde{k}})$ for some K with $K_o \subset K \subsetneq \Gamma$ and a homomorphism $\widetilde{\psi} : K \to S^1$ with $\widetilde{\psi}|_{K_o} = \psi$, $\widetilde{k} \geq k$. Since $(K_o^{\psi,1})$ is a maximal orbit type in the set of all 1-folded twisted orbit types in W, thus $(K_o^{\psi,\widetilde{k}})$ is a maximal orbit type in the set of all $\widetilde{k}$-folded twisted orbit types in W. Consequently, $(K^{\widetilde{\psi},\widetilde{k}}) = (K_o^{\psi,\widetilde{k}})$. Therefore, there exist at least $|\Gamma/K_o|$ different non-constant periodic solutions with the *exact* orbit type $(K_o^{\psi,\widetilde{k}})$. In other words, there exist at least $|\Gamma/K_o|$ different non-constant periodic solutions with the orbit type *at least* $(K_o^{\psi,k})$. □

10.1.3 Computation of $\deg^t$

For simplicity, assume that*

(A5) the operators A and B have only positive eigenvalues.

Consider the complexification V^c of V and the Γ-isotypical decomposition of V^c given by (6.7). Each operator A on V can be extended to a "complexified" operator $A : V^c \to V^c$ given by $A(z \otimes v) := z \otimes Av$ (for which the same notation is used). For each $\mu \in \sigma(A)$, denote by $\widetilde{E}(\mu)$ the eigenspace of μ considered in V^c and call

$$\widetilde{m}_j(\mu) := \frac{\dim\left(\widetilde{E}(\mu) \cap U_j\right)}{\dim \mathcal{U}_j}, \tag{10.10}$$

---

[*] In the case A and B have negative eigenvalues, the argument remains valid for the "positive" parts of $\sigma(A)$ and $\sigma(B)$.

the *$\mathcal{U}_j$-multiplicity* of μ.

Put $A^j := A|_{U_j}$ and

$$\sigma_j^k(A) := \{\mu \in \sigma(A^j) \; : \; k^2 < \mu < (k+1)^2\},$$

thus by the assumption (A4),

$$\sigma(A^j) = \bigcup_{k=0}^{\infty} \sigma_j^k(A).$$

Recall $\mathcal{A}' := \mathcal{A}|_{W'}$, $W' := (W^{S^1})^\perp$. The definition of $\mathcal{A}$ (cf. (10.6)) clearly implies that

$$\begin{aligned}\sigma(\mathcal{A}') &= \left\{1 - \frac{\mu+1}{l^2+1} \; : \; \mu \in \sigma(A), \; l = 1, 2, \dots\right\}\\ &= \left\{1 - \frac{\mu+1}{l^2+1} \; : \; \mu \in \sigma_j^k(A), \; j = 0, 1, \dots, s, \; k = 0, 1, \dots, l = 1, 2, \dots\right\}.\end{aligned}$$

Consequently, the negative spectrum $\sigma_-(\mathcal{A}')$ of $\mathcal{A}'$ can be described by

$$\sigma_-(\mathcal{A}') = \left\{1 - \frac{\mu+1}{l^2+1} \; : \; \mu \in \sigma_j^k(A), \; j = 0, 1, \dots, s, \; k = 0, 1, \dots, l = 1, \dots, k\right\}. \tag{10.11}$$

Moreover, for an eigenvalue $1 - \frac{\mu+1}{l^2+1}$ of $\mathcal{A}'|_{W_l} : W_l \to W_l$, we have

$$m_{j,l}\left(1 - \frac{\mu+1}{l^2+1}\right) = \widetilde{m}_j(\mu), \quad l = 1, 2, \dots \tag{10.12}$$

Therefore, by (10.11)—(10.12), the second component $\deg_0^*$ of $\deg_0$ equals to

$$\begin{aligned}\deg_0^* &= \deg_0^0 * \sum_{\xi \in \sigma_-(\mathcal{A}')} \sum_{j,l} m_{j,l}(\xi) \deg_{\mathcal{V}_{j,l}}\\ &= \deg_0^0 * \sum_{j=0}^{s} \sum_{k=1}^{\infty} \sum_{l=1}^{k} \sum_{\mu \in \sigma_j^k(A)} m_{j,l}(1 - \frac{\mu+1}{l^2+1}) \deg_{\mathcal{V}_{j,l}}\\ &= \deg_0^0 * \sum_{j=0}^{s} \sum_{k=1}^{\infty} \sum_{\mu \in \sigma_j^k(A)} \widetilde{m}_j(\mu) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}.\end{aligned} \tag{10.13}$$

On the other hand, $A^j : U_j \to U_j$ is completely diagonalizable, thus

$$\widetilde{m}_j = \sum_{\mu\in\sigma(A^j)} \widetilde{m}_j(\mu) = \sum_{k=0}^{\infty}\sum_{\mu\in\sigma_j^k(A)} \widetilde{m}_j(\mu). \tag{10.14}$$

Now, by putting

$$\widetilde{m}_j^k(A) := \sum_{\mu\in\sigma_j^k(A)} \widetilde{m}_j(\mu), \tag{10.15}$$

we can simplify (10.13) to the following form:

$$\deg_0^t = \deg_0^0 * \sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^k(A)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}.$$

Notice that (cf. (10.14))

$$\widetilde{m}_j = \sum_{k=1}^{\infty}\widetilde{m}_j^k(A). \tag{10.16}$$

Following the same lines for the operator $\mathcal{B}$, by assumption (A5), one obtains

$$\deg_\infty^t = \deg_\infty^0 * \sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^k(B)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}},$$

and

$$\widetilde{m}_j = \sum_{k=1}^{\infty}\widetilde{m}_j^k(B), \tag{10.17}$$

where

$$\widetilde{m}_j^k(B) := \sum_{\eta\in\sigma_j^k(B)} \widetilde{m}_j(\eta),$$

with $\widetilde{m}_j(\eta)$ being the $\mathcal{U}_j$-isotypical multiplicity of η (cf. (10.10)).

By Theorem 10.1.3, $\deg^0 = 0$, thus $\deg_0^0 = \deg_\infty^0$. Put

$$\deg_o := \deg_0^0 = \deg_\infty^0. \tag{10.18}$$

Therefore, by (10.9),

$$\begin{aligned}
\deg^{t} &= \deg^{t}_{\infty} - \deg^{t}_{0} \\
&= \deg_{o} * \sum_{j=0}^{s}\sum_{k=1}^{\infty}\left((\widetilde{m}_j^k(B) - \widetilde{m}_j^k(A)) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \right) \\
&= \prod_{\mu \in \sigma_-(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} * \sum_{j=0}^{s}\sum_{k=1}^{\infty}\left(\mathfrak{m}_j^k \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \right),
\end{aligned} \tag{10.19}$$

where

$$\mathfrak{m}_j^k := \widetilde{m}_j^k(B) - \widetilde{m}_j^k(A). \tag{10.20}$$

Definition 10.1.4. We call the number $\mathfrak{m}_j^k$ given by (10.20) the *k-th $\mathcal{U}_j$-isotypical compartmental defect number* for the map $\nabla\Psi$, for $j = 0, 1, \dots, s$ and $k = 1, 1, \dots$.

The following lemma describes the possible combinations of the $\mathcal{U}_j$-isotypical compartmental defect numbers $\mathfrak{m}_j^k$, $k = 1, 2, \dots$, subject to conditions (10.16)—(10.17):

Lemma 10.1.5. *Let a, N be positive integers, $(n_k)_{k=1}^N$ and $(m_k)_{k-1}^N$ be two N-part partitions of a, i.e.*

$$a = n_1 + n_2 + \cdots + n_N = m_1 + m_2 + \cdots + m_N,$$

where n_k's and m_k's are non-negative integers. Put

$$b_k := n_k - m_k, \quad k = 1, 2, \dots, N,$$
$$b^+ := \sum_{b_k > 0} b_k, \quad b^- := \sum_{b_k < 0} b_k,$$

where a sum over the empty set is assumed to be 0.

Then $(b_k)_{k=1}^N$ is a partition of 0 with $0 \le b^+ \le a$ and $-a \le b^- \le 0$.

Proof: Assume that $(n_k)_{k=1}^N$ and $(m_k)_{k=1}^N$ are partitions of a, i.e.

$$a = n_1 + n_2 + \cdots + n_N = m_1 + m_2 + \cdots + m_N.$$

Then, clearly, $(b_k)_{k-1}^N = (n_k - m_k)_{k-1}^N$ is a partition of 0 and, by definition, $b^+ \ge 0$, $b^- \le 0$. Moreover, since $n_k \ge 0$ and $m_k \ge 0$ for all k,

$$b^+ = \sum_{b_k>0} b_k = \sum_{n_k>m_k} (n_k - m_k) \le \sum_{n_k>m_k} n_k \le \sum_{k=1}^{N} n_k = a$$

$$b^- = \sum_{b_k<0} b_k = \sum_{n_k<m_k} (n_k - m_k) \ge \sum_{n_k<m_k} (-m_k) \ge -\sum_{k=1}^{N} m_k = -a,$$

which concludes the proof. □

10.1.4 Concrete Existence Results for Selected Symmetries

We present here the computational results for several Γ-representations, where $\Gamma = D_4, D_5, D_6$, S_4 and A_5. Similarly to Subsection 6.3.4, we assume the conditions (R1)—(R2) hold.

By condition (R2),

$$m_i(\mu) = \begin{cases} \dim_{\mathbb{R}} E(\mu)/\dim_{\mathbb{R}} \mathcal{V}_i & i = i_\mu, \\ 0 \quad i \ne i_\mu. \end{cases} \tag{10.21}$$

Also notice that $(\deg_{\mathcal{V}_i})^2 = (G)$ for all i. Put

$$\varepsilon_i = \sum_{\mu\in\sigma(A)} m_{i_\mu}(\mu) \mod 2.$$

Thus,

$$\mathrm{Deg}^o_\Gamma = \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i}.$$

Consequently, the computational formula (10.19) reduces to

$$\deg_1 = \prod_{i=0}^{r} \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i} * \sum_{j=0}^{s} \sum_{k=1}^{\infty} \left(\mathfrak{m}_j^k \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \right). \tag{10.22}$$

Consider the system (10.1) assuming that (A1)—(A5). As the symmetry group Γ take the dihedral groups D_4, D_5, D_6, the octahedral group S_4 and the icosahedral group A_5. We list the computational results in Appendix A4.5.

10.2 Symmetric Autonomous Newtonian System with Degeneracy

In this section, we study the symmetric autonomous Newtonian system (10.1) without assuming the nondegeneracy assumption (A4). In this case, the linearized operator $\mathcal{A}$ (resp. $\mathcal{B}$) at 0 (resp. ∞) may have nontrivial kernel, i.e. the energy functional Ψ defined by (10.3) has degenerate critical points at 0 and/or ∞. As the standard linearization argument fails, the formulae (10.7)–(10.8) are no more valid for the computation of $\deg_{\infty} - \deg_0$. To proceed with this degenerate situation, we need the result called *splitting lemma* proved in [69].

10.2.1 Splitting Lemma

Let G be a compact Lie group and W an (infinite-dimensional) isometric Hilbert G-representation. Consider a C^2-differentiable G-invariant map $\Phi : W \to \mathbb{R}$, which has the following form

$$\Phi(x) = \frac{1}{2}\langle x, x\rangle_W - g(x), \tag{10.23}$$

where $\langle \cdot, \cdot\rangle_W$ denotes the G-invariant inner product on W and $g : W \to \mathbb{R}$ is a G-invariant function satisfying

(B1) $\nabla g : W \to W$ is a G-equivariant compact map.
(B2) For $p \in \{0, \infty\}$, there exists a G-equivariant symmetric compact operator $L_p : W \to W$ and a G-invariant $\eta : W \to \mathbb{R}$ such that $\Phi(x) = \frac{1}{2}\langle(\mathrm{Id} - L_p)x, x\rangle_W + \eta_p(x)$ with $\nabla\eta_p : W \to W$ being a compact map and

$$\|\nabla^2\eta_p(x)\| \to 0, \text{ as } \|x\| \to p.$$

(B3) $0 \in \sigma(\mathrm{Id} - L_p)$, i.e. $p \in \{0, \infty\}$ is a degenerate critical point of Φ.
(B4) $p \in \{0, \infty\}$ is isolated as critical point of Φ.

Notice that (B3) implies that $p = 0$ is a critical point of Φ. We also treat $p = \infty$ as a critical point, with Hesse matrix $\mathrm{Id} - L_\infty$. We call ∞ an *isolated* critical point if $\nabla\Phi^{-1}(0)$ is bounded.

Notation 10.2.1 Denote by $Z_p := \mathrm{Ker}(\mathrm{Id} - L_p)$ and $\mathcal{W}_p := \mathrm{Im}(\mathrm{Id} - L_p)$. Since L_p is a compact operator, we have that $\mathrm{Id} - L_p$ is a Fredholm operator of index zero. Thus, Z_p and $\mathcal{W}_p$ are finite and infinite dimensional G-orthogonal representations, respectively. Also, $\mathrm{Id} - L_p$ being a symmetric linear operator, implies that $W = Z_p \oplus \mathcal{W}_p$ and the operator $\mathcal{Q}_p := (\mathrm{Id} - L_p)|_{\mathcal{W}_p}$ is a G-isomorphism.

The following splitting lemma, which is a simplified version of the theorem proved in [69], is essential for computations of the equivariant degree of $\nabla\Phi$ at 0 and ∞.

Lemma 10.2.2. *(Splitting Lemma) Suppose Φ is of the form (10.23) satisfying (B1)—(B4). Then, for each $p\in\{0,\infty\}$, there exist $\varepsilon_p>0$ and a G-equivariant gradient homotopy $\nabla H_p:[0,1]\times W\to W$ such that*

(i) $\nabla H_0^{-1}(0)\cap(cl(B_{\varepsilon_0}(W))\times[0,1])=\{0\}\times[0,1]$, and $\nabla H_\infty^{-1}(0)\subset cl(B_{\varepsilon_\infty}(W))\times[0,1]$.
(ii) $\nabla H_p(t,\cdot)=\mathrm{Id}-\nabla g_p(t,\cdot)$ for $t\in[0,1]$, where $\nabla g_p:[0,1]\times W\to W$ is a compact map.
(iii) $\nabla H_p(0,\cdot)=\nabla\Phi$, and
(iv) there exists a G-equivariant gradient mapping $\nabla\varphi_p:Z_p\to Z_p$ such that $\nabla H_p(1,(v,w))=(\nabla\varphi_p(v),\mathcal{Q}_p(w))$, for $(v,w)\in Z_p\oplus\mathcal{W}_p$.

Therefore, by the multiplicativity property of gradient equivariant degree, we have (cf. Theorem 5.2.5)

Corollary 10.2.3. *Suppose Φ is of the form (10.23) satisfying (B1)–(B4). Then, for $p\in\{0,\infty\}$, there exist $\varepsilon_p>0$ and a G-equivariant gradient map $\nabla\varphi_p:Z_p\to Z_p$ such that*

$$\nabla_G\text{-deg}\,(\nabla\Phi,B_{\varepsilon_p}(W))=\nabla_G\text{-deg}\,(\nabla\varphi_p,B_{\varepsilon_p}(Z_p))*\nabla_G\text{-deg}\,(\mathcal{Q}_p,B(\mathcal{W}_p)),$$

where Z_p, $\mathcal{W}_p$ and $\mathcal{Q}_p$ are given by Notation 10.2.1.

Remark 10.2.4. Notice that in the case $G=\Gamma\times S^1$ (as usual, we assume Γ is finite), the computational formula (5.28) in Subsection 5.2.2 can be easily extended to the class of G-equivariant gradient compact fields. Indeed, it is well-known that each compact operator has a spectrum either composed of 0 and a finite number of eigenvalues, or it is an infinite sequence of eigenvalues convergent to 0 (which is also in the spectrum). Moreover, every non-zero eigenvalue has a finite multiplicity. Consequently, by compactness assumption (A2), there are only finitely many eigenvalues μ of L_p such that $\mu>1$, which implies that the negative spectrum of $\mathcal{Q}_p=\mathrm{Id}-L_p$ consists of only finitely many eigenvalues, each of which has a finite multiplicity. Therefore, by the suspension property of the gradient degree in the infinite-dimensional case, we have the following analog of formula (5.28), which can be used for the computations of ∇_G-deg$\,(\mathcal{Q}_p,B(\mathcal{W}_p,p))$.

Proposition 10.2.5. *Let $G = \Gamma \times S^1$ for a finite group Γ and let $\mathcal{W}$ be an isometric Hilbert G-representation. Suppose that $\mathcal{Q} : \mathcal{W} \to \mathcal{W}$ is a linear isomorphic G-equivariant gradient compact field. Then,*

$$\begin{aligned}\nabla_G\text{-deg}\,(\mathcal{Q}, B(\mathcal{W})) = \nabla_G\text{-deg}\,(\bar{\mathcal{Q}}, B(\mathcal{W}^{S^1})) - \\ \nabla_G\text{-deg}\,(\bar{\mathcal{Q}}, B(\mathcal{W}^{S^1})) * \sum_{\xi \in \sigma_-(\mathcal{Q}')} \sum_{j,l} m_{j,l}(\xi)\, deg_{\mathcal{V}_{j,l}},\end{aligned}$$

where $\nabla_G\text{-deg}\,(\bar{\mathcal{Q}}, B(\mathcal{W}^{S^1}))$ *is given by*

$$\nabla_G\text{-deg}\,(\bar{\mathcal{Q}}, B(\mathcal{W}^{S^1})) = \prod_{\mu \in \sigma_-(\bar{\mathcal{Q}})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}.$$

10.2.2 Symmetric Autonomous Newtonian Systems with Degeneracy

Consider the symmetric autonomous Newtonian system (10.1) satisfying (A1)—(A3) and the *degeneracy* assumption

(A4') $(\sigma(A) \cup \sigma(B)) \cap \{l^2 \,:\, l = 0, 1, 2, \dots\} \neq \emptyset$.

For simplicity, assume that $\sigma(A)$ (resp. $\sigma(B)$) has a nontrivial intersection with $\{l^2 \,:\, l = 0, 1, 2, \dots\}$, which contains only one element, namely

$$\text{(D)} \begin{cases} \sigma(A) \cap \{l^2 \,:\, l = 0, 1, 2, \dots\} = \{l_0^2\}, \\ \sigma(B) \cap \{l^2 \,:\, l = 0, 1, 2, \dots\} = \{l_\infty^2\}. \end{cases}$$

10.2.3 Reformulation in Functional Spaces

Following the same lines as in Subsection 10.1.1, we reformulate the problem of finding nonstationary periodic solutions of (10.1) to a variational problem of finding nontrivial critical points to the energy functional Ψ defined by (10.3).

By (A1)—(A3) and (D), we are in the setting of Section 10.2.1. Indeed,

$$\Phi(u) = \frac{1}{2}\langle u, u\rangle_{H^1} - \int_0^{2\pi} \widetilde{\varphi}(u(t))dt$$

satisfies (B1)—(B3) for

$$L_0 = j \circ L^{-1} \circ (A + \mathrm{Id}), \tag{10.24}$$
$$L_\infty = j \circ L^{-1} \circ (B + \mathrm{Id}). \tag{10.25}$$

Also, by (A1), the functional Φ satisfies (B4) in the case $l_0 = l_\infty = 0$ in (D) (cf. Lemma 5.2.1. in [69]). In the case $l_p \neq 0$ for some $p \in \{0, \infty\}$, we assume that

(A5) $p \in \{0, \infty\}$ is an isolated critical point of Φ whenever $l_p \neq 0$.

Remark 10.2.6. In general, it is possible that (A5) fails for some $p \in \{0, \infty\}$ with $l_p \neq 0$. However, by an equivariant implicit function theorem argument, it is shown in [69] that in the case (A5) fails, there already exist infinitely many solutions of (10.1) and the minimal period of any solution sufficiently close to the point p is equal to $\frac{2\pi}{l_p}$ (cf. Theorem 5.2.2 in [69]). In particular, (10.1) allows infinitely many nonstationary $\frac{2\pi}{l_p}$-periodic solutions automatically. In this section, we exclude such possibility by assuming (A5).

Therefore, by (A1)—(A3), (D) and (A5), there exist a sufficiently small $\varepsilon > 0$ and large $R > 0$ such that $\deg_0$ and $\deg_\infty$ are well-defined by (10.7)—(10.8). Consequently, Theorem 10.1.3 holds with the assumptions (A1)—(A4) replaced by (A1)—(A3), (D) and (A5). This statement will be referred as Theorem 10.1.3$_d$.

10.2.4 Computations of $\deg_\infty - \deg_0$

To apply Theorem 10.1.3$_d$ for the existence and multiplicity result for the system (10.1) allowing degeneracy assumption, we extend the computations of $\deg_\infty$ and $\deg_0$ using Lemma 10.2.2. Especially, we analyze several possible cases where a nontrivial $(H^{\varphi,l})$-term occurs in $\deg_\infty - \deg_0$, for some dominating orbit type (H^φ). Due to the degeneracy assumption, the value of $\deg_0$ (resp. $\deg_\infty$) is only computable up to an unknown factor. However, to take advantage of Theorem 10.1.3$_d$ (ii), we only need to determine a nontrivial $(H^{\varphi,l})$-term in $\deg_\infty - \deg_0$, i.e. to find a nontrivial $(H^{\varphi,l})$-term in $\deg_\infty$ (resp. $\deg_0$) which does not appear in $\deg_0$ (resp. $\deg_\infty$).

Consider the S^1-isotypical decomposition of W given by (6.31) and take $\mathcal{A}$ and $\mathcal{B}$ defined by (10.6). Then, we have

$$
\begin{aligned}
\mathcal{A}|_{W^{S^1}} &= -A, \quad \mathcal{A}|_{W_l} = \mathrm{Id} - \frac{1}{l^2+1}(A + \mathrm{Id}), \\
\mathcal{B}|_{W^{S^1}} &= -B, \quad \mathcal{B}|_{W_l} = \mathrm{Id} - \frac{1}{l^2+1}(B + \mathrm{Id}).
\end{aligned} \tag{10.26}
$$

We distinguish two degenerate cases for $l_p = 0$ and for $l_p > 0$.

(D_A) $\sigma(A) \cap \{l^2 : l = 0, 1, 2, \dots\} = \{0\}$,
(D'_A) $\sigma(A) \cap \{l^2 : l = 0, 1, 2, \dots\} = \{l_0^2 \neq 0\}$,
(D_B) $\sigma(B) \cap \{l^2 : l = 0, 1, 2, \dots\} = \{0\}$,
(D'_B) $\sigma(B) \cap \{l^2 : l = 0, 1, 2, \dots\} = \{l_\infty^2 \neq 0\}$.

Notice that (cf. (10.26))

$$
\begin{cases}
\mathcal{A} \text{ is a } G\text{-isomorphism on } W^{S^1} & \Leftrightarrow \quad 0 \notin \sigma(A) \\
\mathcal{A} \text{ is a } G\text{-isomorphism on } W_l & \Leftrightarrow \quad l^2 \notin \sigma(A)
\end{cases}, \tag{10.27}
$$

and similar relation holds for $\mathcal{B}$.

Since the computations of $\deg_\infty$ and $\deg_0$ are completely analogous, we only discuss in details the computations of $\deg_0$ assuming (D_A) or (D'_A). A table summarizing the existence/nonexistence of a nontrivial $(H^{\varphi,l})$-term in $\deg_p$, is presented in Theorem 10.2.9, for $p \in \{0, \infty\}$. For completeness, we also include the nondegenerate conditions:

(ND_A) $\sigma(A) \cap \{l^2 : l = 0, 1, 2, \dots\} = \emptyset$,
(ND_B) $\sigma(B) \cap \{l^2 : l = 0, 1, 2, \dots\} = \emptyset$.

By Corollary 10.2.3, there exists $\varepsilon > 0$ and a G-equivariant gradient map $\nabla\varphi_0 : Z_0 \to Z_0$ such that

$$
\deg_0 = \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * \nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0)),
$$

where $\nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0))$ can be computed by (cf. Proposition 10.2.5)

$$
\begin{aligned}
&\nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0)) \\
&= \prod_{\mu \in \sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} \\
&- \prod_{\mu \in \sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} * \sum_{\xi \in \sigma_-(\mathcal{A}|_{\mathcal{W}_0'})} \sum_{j,l} m_{j,l}(\xi) \deg_{\mathcal{V}_{j,l}}.
\end{aligned}
$$

To simplify the notations, put

$$\deg^0_{\mathcal{A}} := \prod_{\mu\in\sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}, \tag{10.28}$$

$$\deg^l_{\mathcal{A}} := \deg^0_{\mathcal{A}} * \sum_{\xi\in\sigma_-(\mathcal{A}|_{\mathcal{W}_0'})} \sum_{j,l} m_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}}. \tag{10.29}$$

Then, we have

$$\deg_0 = \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * (\deg^0_{\mathcal{A}} - \deg^l_{\mathcal{A}}). \tag{10.30}$$

We simplify the formulae (10.28)—(10.29), under different assumptions (D_A), (D'_A) and (ND_A) respectively.

Case ($\mathbf{D}_A$): Under the assumption (D_A), $\mathcal{A}|_{W_l}$ is a linear G-isomorphism of W_l for each $l \in \{1, 2, \dots\}$, and $Z_0 = \operatorname{Ker}\mathcal{A} = \operatorname{Ker} A \subset W^{S^1}$ (cf. (10.27)). Thus,

$$\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \in A_0(G). \tag{10.31}$$

Therefore,

$$\begin{aligned}\deg_0 &= \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * (\deg^0_{\mathcal{A}} - \deg^l_{\mathcal{A}})\\ &= \underbrace{\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * \deg^0_{\mathcal{A}}}_{\in A_0(G)} - \underbrace{\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * \deg^l_{\mathcal{A}}}_{\in A_1(G)},\end{aligned}$$

where $-\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * \deg^l_{\mathcal{A}}$ is the part that may contribute a non-trivial $(H^{\varphi,l})$-term to $\deg_0$.

Since $\mathcal{W}_0^{S^1} = \operatorname{Im}(A)$, we have that $\sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}}) = \sigma_+(A)$ (cf. (10.26)). To interpret the formula (10.29), it is sufficient to observe that

$$\xi \in \sigma_-(\mathcal{A}|_{\mathcal{W}_0'}) \iff \xi = 1 - \frac{\mu+1}{l^2+1},\ \mu > l^2,\ \text{for } \mu\in\sigma(A),\ l\in\{1,2,\dots\},$$

and

$$m_{j,l}(\xi) = \widetilde{m}_j(\mu),$$

where $\widetilde{m}_j(\mu)$ is the $\mathcal{U}_j$-multiplicity of μ.

Let $\widetilde{m}_j^k(A)$ by defined by (10.15). It can be directly verified that (cf. (10.13))

$$\sum_{\xi\in\sigma_-(\mathcal{A}')}\sum_{j,l} m_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}} = \sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^k(A)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}.$$

Therefore, the formulae (10.28)–(10.29) reduce to

$$\deg^0_{\mathcal{A}} = \prod_{\mu\in\sigma_+(A)}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)},$$

$$\deg^l_{\mathcal{A}} = \deg^0_{\mathcal{A}} * \sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^k(A)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}.$$

Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W. We introduce the following conditions:

(Y1) $\deg^l_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term, and $Z_0 = \operatorname{Ker} A$ is such that

$$\begin{cases}(Z_0)^{\Gamma} = \{0\}\\ (\widetilde{H}\times S^1)\notin \mathcal{J}(Z_0) \text{ for any } (\widetilde{H}) \; s.t. \; (H)\le(\widetilde{H})<(\Gamma).\end{cases}$$

(N1) $\deg^l_{\mathcal{A}}$ does not contain a nontrivial $(H^{\varphi,l})$-term.

Proposition 10.2.7. *Let $\varphi : V \to \mathbb{R}$ be a Γ-invariant C^2-differentiable map satisfying (A1)–(A3) and (D_A). Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W. Then,*

(i) Under the assumption (Y1), there exists a $(H^{\varphi,l})$-term with a non-zero coefficient in $\deg_0$;

(ii) Under the assumption (N1), there is no $(H^{\varphi,l})$-term with non-zero coefficient in $\deg_0$.

Proof: (i). By $(Z_0)^{\Gamma} = \{0\}$ and $Z_0 \subset W^{S^1}$, we have that $(Z_0)^G = \{0\}$, and

$$\nabla_G\text{-deg}\,(\nabla\varphi, B_\varepsilon(Z_0)) = (G) + a_0 \in A_0(G), \tag{10.32}$$

for some $a_0 \in A_0(G)$, which does not contain nontrivial (G)-term. Substituting (10.32) in (10.30), we obtain

$$\begin{aligned}\deg_0 &= \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * \nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0))\\ &= \big((G)+a_0\big) * (\deg^0_{\mathcal{A}} - \deg^l_{\mathcal{A}})\\ &= \deg^0_{\mathcal{A}} - \deg^l_{\mathcal{A}} + a_0 * \deg^0_{\mathcal{A}} - a_0 * \deg^l_{\mathcal{A}}\\ &= \underbrace{\deg^0_{\mathcal{A}} + a_0 * \deg^0_{\mathcal{A}}}_{\in A_0(G)}\underbrace{-\deg^l_{\mathcal{A}} - a_0 * \deg^l_{\mathcal{A}}}_{\in A_1(G)},\end{aligned}$$

Since $\deg^{\,t}_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term, to conclude that $\deg_0$ also contains this $(H^{\varphi,l})$-term (with an opposite sign), it suffices to eliminate the possibility that

$$a_0 * \deg^{\,t}_{\mathcal{A}} = -(H^{\varphi,l}) + \text{rest}.$$

By the maximality of (H^{φ}), this would only happen if a_0 contains a nontrivial $(\widetilde{H} \times S^1)$-term for some $(\widetilde{H}) \geq (H)$. Also notice that $(\widetilde{H}) < (\Gamma)$, since a_0 does not contain (G)-term. By the assumption that such a $(\widetilde{H} \times S^1)$ does not occur in $\mathcal{J}(Z_0)$, it is impossible for a_0 to contain such a nontrivial $(\widetilde{H} \times S^1)$-term, so the statement follows.

(ii). It is clear that if $\deg^{\,t}_{\mathcal{A}}$ has no nontrivial $(H^{\varphi,l})$-term, $\deg_0$ does not permit one. □

Case $(\mathbf{D}'_A)$: Under the assumption (D'_A), $\mathcal{A}$ is a linear G-isomorphism when restricted to the S^1-isotypical components W^{S^1} and each W_l, for $l \neq l_0$ (cf. (10.26)). Indeed,

$$Z_0 = \operatorname{Ker} \mathcal{A} \subset W_{l_0}.$$

In particular, $(Z_0)^{S^1} = \{0\}$ and

$$\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) = (G) + a_1, \quad \text{for } a_1 \in A_1(G). \tag{10.33}$$

Substituting (10.33) in (10.30), we obtain

$$\begin{aligned}
\deg_0 &= \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) * \nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0)) \\
&= \big((G) + a_1\big) * (\deg^{\,0}_{\mathcal{A}} - \deg^{\,t}_{\mathcal{A}}) \\
&= \deg^{\,0}_{\mathcal{A}} - \deg^{\,t}_{\mathcal{A}} + a_1 * \deg^{\,0}_{\mathcal{A}} - a_1 * \deg^{\,t}_{\mathcal{A}} \\
&= \underbrace{\deg^{\,0}_{\mathcal{A}}}_{\in A_0(G)} \underbrace{- \deg^{\,t}_{\mathcal{A}} + a_1 * \deg^{\,0}_{\mathcal{A}}}_{\in A_1(G)},
\end{aligned}$$

where the last equality uses the fact that $a_1 * \deg^{\,t}_{\mathcal{A}} = 0$, since $a_1, \deg^{\,t}_{\mathcal{A}} \in A_1(G)$ (cf. Proposition 5.1.14).

Moreover, we have

$$\begin{aligned}
\deg^{\,t}_{\mathcal{A}} &= \deg^{\,0}_{\mathcal{A}} * \sum_{\xi \in \sigma_-(\mathcal{A}')} \sum_{j=0}^{s} \sum_{l=1}^{\infty} m_{j,l}(\xi) \deg_{\mathcal{V}_{j,l}} \\
&= \deg^{\,0}_{\mathcal{A}} * \Big(\sum_{j=0}^{s} \sum_{k=1}^{\infty} \widetilde{m}^k_j(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s} \widetilde{m}_j(l_0^2) \sum_{l=1}^{l_0-1} \deg_{\mathcal{V}_{j,l}} \Big),
\end{aligned} \tag{10.34}$$

where it is clear that

$$\deg^{0}_{\mathcal{A}} = \prod_{\mu\in\sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}. \tag{10.35}$$

We introduce the following conditions:

(Y2) $\deg^{t}_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term, and $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$.
(N2) $\deg^{t}_{\mathcal{A}}$ does not contain a nontrivial $(H^{\varphi,l})$-term and $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$.

Proposition 10.2.8. *Let $\varphi : V \to \mathbb{R}$ be a Γ-invariant C^2-differentiable map satisfying (A1)–(A3), (D'_A) and (A5). Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W.*

(i) Under the assumption (Y2), there exists a $(H^{\varphi,l})$-term with non-zero coefficient in $\deg_0$;
(ii) Under the assumption (N2), there is no $(H^{\varphi,l})$-term with non-zero coefficient in $\deg_0$.

Proof: (i). By (Y2), $\deg^{t}_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term. It is sufficient to show that $a_1 * \deg^{0}_{\mathcal{A}}$ does not contain any $-(H^{\varphi,l})$-term so that a cancelation does not occur. But $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$, which implies that a_1 has no nontrivial $(H^{\varphi,l})$-term. Thus, by maximality of $(H^{\varphi,l})$, $a_1 * \deg^{0}_{\mathcal{A}}$ contains no $(H^{\varphi,l})$-term. Therefore, it follows that there exists a $(H^{\varphi,l})$-term with non-zero coefficient in $\deg_0$.

(ii). Similar proof as in (i). By (N2), $\deg^{t}_{\mathcal{A}}$ contains no nontrivial $(H^{\varphi,l})$-term. It is sufficient to show that $a_1 * \deg^{0}_{\mathcal{A}}$ does not contain any $-(H^{\varphi,l})$-term, which is again the case by the condition $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$. □

Case (ND$_A$): Under the nondegeneracy assumption (ND$_A$), $\mathcal{A}$ is a linear G-isomorphism of W. Thus, the complete value of $\deg_0$ can be obtained (cf. Subsection 10.1.3). Then, it makes sense to formulate the following conditions:

(Y) $\deg_0$ contains a nontrivial $(H^{\varphi,l})$-term,
(N) $\deg_0$ does not contain any nontrivial $(H^{\varphi,l})$-term.

Theorem 10.2.9. *Let $\varphi : V \to \mathbb{R}$ be a Γ-invariant C^2-differentiable map satisfying (A1)–(A3) and (A5). Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W. Then, we have the Table 10.2.9 summarizing the sufficient conditions of existence and nonexistence of a nontrivial $(H^{\varphi,l})$-term in $\deg_p$, for $p \in \{0, \infty\}$ (where the conditions (Y1'), (Y'), (N1'), (N2') and (N') of $\mathcal{B}$ are the counterparts of those of $\mathcal{A}$).*

	$\deg_0$	$\deg_\infty$
Existence of $(H^{\varphi,l})$	$(D_A)+(Y1)$	$(D_B)+(Y1')$
	$(D'_A)+(Y2)$	$(D'_B)+(Y2')$
	$(ND_A)+(Y)$	$(ND_B)+(Y')$
Nonexistence of $(H^{\varphi,l})$	$(D_A)+(N1)$	$(H4_0)+(N1')$
	$(D'_A)+(N2)$	$(D'_B)+(N2')$
	$(ND_A)+(N)$	$(ND_B)+(N')$

Table 10.1. Existence and Nonexistence of $(H^{\varphi,l})$-term in $\deg_p$.

Proof: This is an immediate consequence of Propositions 10.2.7–10.2.8. □

Corollary 10.2.10. *Let $\varphi : V \to \mathbb{R}$ be a Γ-invariant C^2-differentiable map satisfying (A1)—(A3) and (A5). Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W. Then, we have a nontrivial $(H^{\varphi,l})$-term in* $\deg_\infty - \deg_0$, *if the conditions in the Table 10.2.9 are satisfied diagonally, i.e. one of the existence conditions for* $\deg_0$ *with one of the nonexistence conditions for* $\deg_\infty$ *or vice versa.*

10.2.5 Computational Examples

We present the computational examples for $\Gamma = D_n$ and $V = \mathbb{R}^n$ for $n = 6, 8, 10, 12$. Consider the potential $\varphi : V \to \mathbb{R}$ satisfying (A1)—(A3) with the matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & \dots & 0 & d \\ d & c & d & 0 & \dots & 0 & 0 \\ 0 & d & c & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d & 0 & 0 & 0 & \dots & d & c \end{bmatrix}.$$

To obtain φ satisfying the above properties, one can define for example $\varphi : V \to \mathbb{R}$ by $\varphi(x) := \frac{1}{2}\langle Bx, x\rangle - \frac{1}{\sqrt{\langle (A-B)x,x\rangle + a}}$, for certain $a > 0$. A similar computational example can be found in [69]. We also assume (A5) in all the computational examples. The degeneracy assumptions are listed in Table 10.2.

Γ	$\deg_0$	$\deg_\infty$
D_6	(D$_A$)+(Y1)	(D$_B$)+(N1')
D_8	(D$_A$)+(Y1)	(D$'_B$)+(N2')
D_{10}	(D$'_A$)+(N2)	(D$_B$)+(Y1')
D_{12}	(D$'_A$)+(N2)	(D$'_B$)+(Y2')

Table 10.2. Summary of the assumptions in the computational examples.

10.3 $\mathcal{O}(2)$-Symmetric Elliptic Equation with Periodic-Dirichlet Mixed Boundaries

Suppose that $\mathcal{O} \subset \mathbb{R}^2 \simeq \mathbb{C}$ is a unite disc and take $\Omega := (0, 2\pi) \times \mathcal{O}$. Consider the following elliptic periodic-Dirichlet BVP

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \Delta_x u(x) = f(u(t,x)), & x \in \Omega \\ u(t,x) = 0 \quad \text{a.e. for} \quad x \in \partial\mathcal{O},\ t \in (0, 2\pi), \\ u(0,x) = u(2\pi, x) \quad \text{a.e. for} \quad x \in \mathcal{O}, \\ \frac{\partial u}{\partial t}(0,x) = \frac{\partial u}{\partial t}(2\pi, x) \quad \text{a.e. for} \quad x \in \mathcal{O}, \end{cases} \tag{10.36}$$

where $(t, x) \in (0, 2\pi) \times \mathcal{O}$, $u \in H^2(\Omega; \mathbb{R})$, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying the conditions:

(B1) $f(0) = 0$ and $f'(0) = a > 0$;
(B2) f is asymptotically linear at infinity, i.e. there exists $b \in \mathbb{R}$ such that

$$\lim_{|t| \to \infty} \frac{f(t) - bt}{t} = 0. \tag{10.37}$$

Consider the Laplace operator $-\Delta_x$ on $\mathcal{O}$ with the Dirichlet boundary condition. Then, the operator $-\Delta_x$ has the spectrum

$$\sigma(-\Delta_x) := \{\mu_{k,j} : \mu_{k,j} = z_{k,j}^2,\ \ k = 1, 2, \dots,\ \ j = 0, 1, 2, \dots, J_j(z_{k,j}) = 0\},$$

where $z_{k,j}$ denotes the k-th zero of the j-th Bessel function J_j. The corresponding to $\mu_{j,k}$ eigenfunctions (expressed in polar coordinates) are; for $j = 0$

$$\varphi_{k,0}(r) := J_0(\sqrt{\mu_{k,j}}\, r),$$

and for $j > 0$,

$$\varphi^c_{k,j}(r,\theta) := J_j(\sqrt{\mu_{k,j}}r)\cos(j\theta), \quad \varphi^s_{k,j}(r,\theta) := J_j(\sqrt{\mu_{k,j}}r)\sin(j\theta).$$

The space $\mathrm{span}\{\varphi^c_{k,j}, \varphi^s_{k,j}\}$ is equivalent to the j-th irreducible $O(2)$-representation $\mathcal{V}_j$ ($j > 0$), and the space $\mathrm{span}\{\varphi_{k,0}\}$ i equivalent to the trivial irreducible $O(2)$-representation $\mathcal{V}_0$. We need additional assumptions

(B3) $a, b \notin \{l^2 + \mu_{k,j},\ \mu_{k,j} \in \sigma(-\Delta_x),\ l = 0, 1, 2, \dots\}$.

(B4) The system

$$\begin{cases} -\Delta_x u = f(u), \\ u|_{\partial\mathcal{O}} = 0, \end{cases} \tag{10.38}$$

has a unique solution $u \equiv 0$.

10.3.1 Setting in Functional Spaces

By using the standard identification $\mathbb{R}/2\pi \simeq S^1$ we can assume that $\Omega := S^1 \times \mathcal{O}$ and that $\partial\Omega = S^1 \times S^1$. We put $W := H^1_0(\Omega) := \{u \in H^1(\Omega; \mathbb{R}) : u|_{\partial\Omega} \equiv 0\}$, which is a Hilbert G-representation for $G = O(2) \times S^1$, with the inner product

$$\langle u, v \rangle := \int_\Omega \nabla u(t) \cdot \nabla v(t)\, dt.$$

Associate to the problem (10.36) a functional $\Psi : \mathbb{R} \oplus W \to \mathbb{R}$ given by

$$\Psi(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(u(x))dx,$$

where $F(y) := \int_0^y f(t)dt$, and define $J : W \to \mathbb{R}$ by

$$J(u) := \int_\Omega F(u(x))dx.$$

Since f is a C^1-function satisfying (B1), J is of class C^1 and for $h \in W$,

$$DJ(u)h = \int_\Omega f(u(x))h(x)dx.$$

Thus, Ψ is also C^1-differentiable with respect to u and

$$D_u\Psi(u)h = \int_\Omega \nabla u(x)\nabla h(x)dx - DJ(u)h, \quad h \in W.$$

Consequently, by the standard argument, if $D_u\Psi(\lambda, u) \equiv 0$, then u is a solution to (10.36). In particular,

$$\nabla_u \Psi(u) = 0 \iff u \text{ is a solution to (10.36)},$$

where

$$\nabla_u \Psi(u) = u - \nabla J(u). \tag{10.39}$$

To determine ∇J, we introduce the following operators (cf. Figure 10.1)

$$\begin{gathered} j : H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad j(u) = u, \\ N_f : L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega), \quad N_f(u)(x) = f(u(x)), \end{gathered}$$

and rewrite $DJ(u) : W \to \mathbb{R}$ as

$$DJ(u)h = \int_\Omega N_f(j(u))(x)h(x)dx. \tag{10.40}$$

$$\begin{tikzcd} H_0^1(\Omega) \arrow[rrr, "\nabla J"] \arrow[dr, "j"'] & & & H_0^1(\Omega) \\ & L^p(\Omega) \arrow[r, "N_f"] & L^{\frac{p}{p-1}}(\Omega) \arrow[ur, "R"'] & \end{tikzcd}$$

Fig. 10.1. Composition diagram for ∇J

It is known that the inclusion j is a compact operator (since f is asymptotically linear, it satisfies $|f(t)| \leq A + B|t|$ for some constants A and B, thus the usual condition $p < \frac{2n}{n-2}$, with $p = 1$ and $n = 3$ is satisfied) and N_f is C^1-differentiable. Thus,

$$\nabla N_f(0) = f'(0)\,\mathrm{Id}. \tag{10.41}$$

Denote by $(H_0^1(\Omega))^*$ the dual space of $H_0^1(\Omega)$ and $\iota : (H_0^1(\Omega))^* \to H_0^1(\Omega)$ the isomorphism given by the Riesz representation theorem. Let $\tau : L^{\frac{p}{p-1}}(\Omega) \to (H_0^1(\Omega))^*$ be a (continuous) map defined by

$$\tau(\psi)(v) := \int_{\Omega} \psi(x)v(x)dx, \quad \psi \in L^{\frac{p}{p-1}}(\Omega), \quad v \in H_0^1(\Omega),$$

and $R : L^{\frac{p}{p-1}}(\Omega) \to H_0^1(\Omega)$ defined by $R := \iota \circ \tau$. Then, R is the inverse of the Laplacian $-\triangle$, i.e. $R\varphi$ is the weak solution to the problem

$$\begin{cases} -\triangle u(t,x) = \varphi, & (t,x) \in \Omega \\ u|_{\partial\Omega} \equiv 0, \end{cases}$$

where $\triangle := \frac{\partial^2}{\partial t^2} + \triangle_x$, or equivalently,

$$\langle R\varphi, h\rangle_{H_0^1(\Omega)} = \int_{\Omega} \varphi(x)h(x)dx, \quad \forall h \in H_0^1(\Omega).$$

In particular, if $\varphi = N_f \circ j(u)$, then

$$\langle R \circ N_f \circ j(u), h\rangle_{H_0^1(\Omega)} = \int_{\Omega} N_f(j(u))(x)h(x)dx.$$

Taking into account (10.40), we obtain

$$\langle R \circ N_f \circ j(u), h\rangle_{H_0^1} = DJ(u)h, \quad h \in W,$$

i.e.

$$\nabla J(u) = R \circ N_f \circ j(u).$$

Therefore (cf. (10.39)),

$$\mathfrak{F}(u) := \nabla\Psi(u) = u - R \circ N_f \circ j(u), \quad u \in W,$$

is a completely continuous $O(2) \times S^1$-equivariant gradient field on W. Then the problem (10.36) is equivalent to the equation

$$\mathfrak{F}(u) = 0. \tag{10.42}$$

10.3.2 Example of a Function f Satisfying (B1)—(B4)

A similar functional setting can be established for the boundary problem (10.38), namely we can reformulate it as the equation

$$\mathfrak{F}_x(u) = 0, \quad u \in H_0^1(\mathcal{O}),$$

where

$$\mathfrak{F}_x(u) := \nabla\Psi_x(u) = u - R_x \circ N_f \circ j(u),$$

with R_x being the inverse of the Laplacian $-\Delta_x$. It s possible to construct a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions (B1)—(B4). We can choose two numbers $0 < a < b$ such that $[a,b] \cap \sigma(-\Delta_x) = \emptyset$ and

$$b - a < \frac{|\mu_{k_o} - b|\mu_1}{\mu_{k_o}}, \quad \frac{|\mu_{k_o} - b|}{\mu_{k_o}} := \max\{\frac{|\mu_k - b|}{\mu_k} : \mu_k \in \sigma(-\Delta_x)\},$$

and put

$$f(u) = bu - (b-a)\frac{u}{1+u^2}, \quad u \in \mathbb{R}.$$

More generally, assume that f is an asymptotically linear function satisfying the conditions (B1)—(B3) and such that $\eta := \max\{|f'(u)| : u \in \mathbb{R}\}$ is such that

$$\eta < \frac{|\mu_{k_o} - b|\mu_1}{\mu_{k_o}}, \quad \frac{|\mu_{k_o} - b|}{\mu_{k_o}} := \max\left\{\frac{|\mu_k - b|}{\mu_k} : \mu_k \in \sigma(-\Delta_x)\right\}. \tag{10.43}$$

Then, clearly, $b - a \le \eta$.

Proposition 10.3.1. *Under the above assumptions the boundary problem (10.38) has a unique solution* $u \equiv 0$.

Proof: Let us observe that under the condition (10.43), the derivative $D\mathfrak{F}_x : H_0^1(\mathcal{O}) \to H_0^1(\mathcal{O})$ is an isomorphism for all $u \in H_0^1(\mathcal{O})$. Indeed,

$$D\mathfrak{F}_x(u)(v) = v - bR_x j(v) - R_x[N_{f'(u)} - b\mathrm{Id}]j(v), \quad N_{f'(u)}j(v)(x) := f'(u(x))v(x).$$

Put

$$A := \mathrm{Id} - bR_x \circ j, \quad B := R_x[N_{f'(u)} - b\mathrm{Id}] \circ j.$$

Then $D\mathfrak{F}_x(u) = A - B$, and we have (by (B3) that A is invertible with $\|A^{-1}\| = \frac{|\mu_{k_o} - b|}{\mu_{k_o}}$ and $\|B\| \le \eta\|R_x\| = \mu_1\eta$. Then the operator

$$D\mathfrak{F}_x(u) = A - B = A(\mathrm{Id} - A^{-1}B)$$

is invertible if $\|A^{-1}B\| < 1$. But,

$$\|A^{-1}B\| \le \||A^{-1}\||B\| \le \frac{|\mu_{k_o} - b|}{\mu_{k_o}}\mu_1\eta < 1.$$

Consequently, every solution $u \in H_0^1(\mathcal{O})$ to the problem (10.38) (i.e. $\mathfrak{F}_x(u) = 0$) is a regular point of $\mathfrak{F}_x$ and consequently, it has to be an isolated solution. Since (10.38) is $O(2)$-symmetric, it follows that the isotropy of u is $O(2)$, i.e. u is a radial function on $\mathcal{O}$ (which can be detected using Leray-Schauder degree). Since $D\mathfrak{F}_x(0)$, $D\mathfrak{F}_x(\infty) : H_0^1(\mathcal{O}) \to H_0^1(\mathcal{O})$ are isomorphisms and $\mathfrak{F}_x$ is a completely continuous vector field on $H_0^1(\mathcal{O})$, there can only be finitely many solutions to the equation (10.38), and for every solution u the Leray-Schauder degree $\mathrm{Deg}(\mathfrak{F}_x, B_u)$ is well defined on an isolating neighborhood B_u of u. By using the linearization of $\mathfrak{F}_x$ on B_u, by the condition (10.43),

$$\mathrm{Deg}\left(\mathfrak{F}_x, B_u\right) = \mathrm{Deg}\left(D\mathfrak{F}_x(0), B_1(0)\right) = \mathrm{Deg}\left(D\mathfrak{F}_x(\infty), B_1(0)\right) \neq 0.$$

Therefore, by the additivity property of the Leray-Schauder degree, there can only be one solution $u \equiv 0$. □

10.3.3 Equivariant Invariant and Isotypical Decomposition of *W*

By assumption (B3), there exists R, $\varepsilon > 0$ such that $u = 0$ is the only solution to the equation (10.42) in $\overline{B_\varepsilon(0)} \subset W$, and (10.42) has no solutions $u \in W$ such hat $\|u\| \geq R$. We define the *equivariant invariant* ω for the problem (10.36) by

$$\omega := \deg_0 - \deg_\infty, \tag{10.44}$$

where

$$\deg_0 := \nabla_{O(2)\times S^1}\text{-}\deg\left(\mathfrak{F}, B_\varepsilon(0)\right), \quad \deg_\infty := \nabla_{O(2)\times S^1}\text{-}\deg\left(\mathfrak{F}, B_R(0)\right).$$

The spectrum σ of $-\Delta$ on Ω (with the boundary conditions (10.36)) is

$$\sigma = \{\lambda_{k,j,l} : \lambda_{k,j,l} := l^2 + \mu_{k,j},\ \ \mu_{k,j} \in \sigma(-\Delta_x),\ l = 0, 1, 2, \dots\}.$$

Denote by $E_{k,j,l}$ the eigenspace of $-\Delta$ in W corresponding to the eigenvalue $\lambda_{k,j,l}$. Observe that $E_{k,j,l}$, for $j, l > 0$ is equivalent to the irreducible orthogonal $O(2) \times S^1$-representation $\mathcal{V}_{j,l}$ and

$$E_{k,j,l} = \mathrm{span}\{\cos lt \cdot \varphi^c_{k,l}(x), \cos lt \cdot \varphi^s_{k,j}(x), \sin lt \cdot \varphi^c_{k,j}(x), \sin lt \cdot \varphi^s_{k,j}(x)\}.$$

If $j = 0$ and $l > 0$, then

$$E_{k,0,l} = \mathrm{span}\{\cos lt \cdot \varphi_{k,0}(x), \sin lt \cdot \varphi_{k,0}(x)\},$$

and it is equivalent to the irreducible orthogonal $O(2) \times S^1$-representation $\mathcal{V}_{0,l}$. If $j > 0$ and $l = 0$,

$$E_{k,j,0} = \operatorname{span}\{\varphi^c_{k,j}(x), \varphi^s_{k,j}(x)\} \simeq \mathcal{V}_j,$$

and for $j = l = 0$, we have that

$$E_{k,0,0} = \operatorname{span}\{\varphi_{k,0}(x)\},$$

is equivalent to the trivial $O(2) \times S^1$-representation $\mathcal{V}_0$. The $O(2) \times S^1$-isotypical components of the space W are

$$W_{j,l} := \overline{\bigoplus_k E_{k,j,l}}, \quad j, l = 0, 1, 2, \dots$$

10.3.4 Computation of the Equivariant Invariant

Assume that $0 < a < b$ and that the following condition holds:

(B5) there exists (k_o, j_o, l_o), $l_o \geq 1$, such that

$$\sigma(-\triangle) \cap (a, b) = \{\lambda_{k_o, j_o, l_o}\}$$

Put $p = 0$ or ∞ and denote by σ^-_p the negative spectrum of $D\mathfrak{F}(p)$, i.e.

$$\begin{aligned}\sigma^-_0 &:= \{\lambda \in \sigma(D\mathfrak{F}(0)) : \lambda < 0\} \\ &= \{\lambda = 1 - \frac{a}{\lambda_{k,j,l}} \ : \ \lambda_{k,j,l} < a\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma^-_\infty &:= \{\lambda \in \sigma(D\mathfrak{F}(\infty)) : \lambda < 0\} \\ &= \{\lambda = 1 - \frac{b}{\lambda_{k,j,l}} \ : \ \lambda_{k,j,l} < b\}.\end{aligned}$$

By assumption (B5), $\sigma^-_\infty = \sigma^-_0 \cup \{\lambda_o\}$, $\lambda_o := \lambda_{k_o, j_o, l_o}$. The linear operator $D\mathfrak{F}(p)$ is G-homotopic (in the class of gradient maps) to

$$A_p = (-\mathrm{Id}\,) \times \mathrm{Id} \, : E_p \oplus E^\perp_p \to E_p \oplus E^\perp_p, \quad E_p := \bigoplus_{\lambda_{k,j,l} \in \sigma^-_p} E_{k,j,l},$$

and consequently

$$\begin{aligned}\deg_p = \nabla_G\text{-deg}\,(A_p, B_1(0)) &= \prod_{\lambda\in\sigma_p^-} \nabla_G\text{-deg}\,(-\mathrm{Id}\,, B_1(E_{k,j,l}))\\ &= \prod_{\lambda\in\sigma_p^-} \mathrm{Deg}_{\mathcal{V}_{j,l}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\omega = \deg_0 - \deg_\infty &= \prod_{\lambda\in\sigma_0^-} \mathrm{Deg}_{\mathcal{V}_{j,l}} * \Big((G) - \mathrm{Deg}_{\mathcal{V}_{j_o,l_o}}\Big)\\ &= \prod_{\lambda_{k,j,l}<a} \mathrm{Deg}_{\mathcal{V}_{j,l}} * \Big((SO(2)^{\varphi_{j_o},l_o}) + (D_{2j_o}^{d,l_o}) - (\mathbb{Z}_{2j_o}^{d,l_o})\Big). \qquad (10.45)\end{aligned}$$

Notice that by Remark 5.2.22, the element $\mathfrak{a} := \prod_{\lambda_{k,j,l}\in\sigma_0^-} \mathrm{Deg}_{\mathcal{V}_{j,l}}$ is invertible, therefore $\omega \neq 0$. Moreover, by using the multiplication table for $U(O(2)\times S^1)$ and the list of basic gradient degrees for irreducible $O(2)\times S^1$-representations, one can easily conclude that

$$\mathfrak{a} * (SO(2)^{\varphi_{j_o},l_o}) = (SO(2)^{\varphi_{j_o},l_o}) + x^*, \qquad \text{and} \qquad \mathfrak{a} * (D_{2j_o}^{d,l_o}) = \pm(D_{2j_o}^{d,l_o}) + y^*,$$

where x^* and y^* denotes the other terms in $U(G)$, which do not contain $(SO(2)^{\varphi_{j_o},l_o})$ and $(D_{2j_o}^{d,l_o})$.

Consequently, we can formulate the following existence result

Theorem 10.3.2. *Under the assumptions (B1)—(B4) the equation (10.36) has at least two $O(2)\times S^1$-orbits of non-trivial t-periodic solutions with the orbit types at least $(SO(2)^{\varphi_{j_o},l_o})$ and $(D_{2j_o}^{d,l_o})$ respectively.*

Let us point out that the periodic solutions corresponding to the orbit types $(SO(2)^j)$ are commonly called *rotating waves* or *spiral vortices* while those with he orbit type (D_{2j}^d) are called *ribbons* or *stationary waves*. Therefore, it seems appropriate to call the t-periodic solutions with the orbit type $(SO(2)^{\varphi_{j_o},l_o})$ the l_o-folded *rotating waves* or *spiral vortices* and those with the orbit type $(D_{2j_o}^{d,l_o})$ the l_o-folded *ribbons* or *stationary waves.*

Example 10.3.3. To supply the numbers a and b satisfying (B5), we need to have an increasing ordered sequence of the values $\lambda_{k,j,l}$ on the real line $\mathbb{R}$. Recall that $\lambda_{k,j,l} = l^2 + z_{k,j}^2$, where $z_{k,j}$ is the k-th zero of the j-th Bessel function . By calling the Maple© command `evalf((BesselJZeros(j,k))^ 2)`, we obtain

j	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$\dots$
0	5.78	30.47	74.88	139.04	$\dots$
1	14.68	49.22	103.50	177.52	$\dots$
2	26.37	70.85	135.02	218.92	$\dots$
3	40.71	95.28	169.40	263.20	$\dots$
4	57.58	122.43	206.57	310.32	$\dots$
5	76.94	152.24	246.50	360.25	$\dots$
6	98.73	184.67	289.13	412.93	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 10.3. Approximate values of $z_{k,j}^2$, where the zigzag line indicates the first 12 smallest values.

(k, j)	$z_{k,j}^2$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$	$l = 7$	$l = 8$	$\dots$
$(1, 0)$	5.78	6.78	9.78	14.78	21.78	30.78	41.78	54.78	69.78	$\dots$
$(1, 1)$	14.68	15.68	18.68	23.68	30.68	39.68	50.68	63.68	78.68	$\dots$
$(1, 2)$	26.37	27.37	30.37	35.37	42.37	51.37	62.37	75.37	90.37	$\dots$
$(2, 0)$	30.47	31.47	34.47	39.47	46.47	55.47	66.47	79.47	94.47	$\dots$
$(1, 3)$	40.71	41.71	44.71	49.71	56.71	65.71	76.71	89.71	104.71	$\dots$
$(2, 1)$	49.22	50.22	53.22	58.22	65.22	74.22	85.22	98.22	113.22	$\dots$
$(1, 4)$	57.58	58.58	61.58	66.58	73.58	82.58	93.58	106.58	121.58	$\dots$
$(2, 2)$	70.85	71.85	74.85	79.85	86.85	95.85	106.85	119.85	134.85	$\dots$
$(3, 0)$	74.88	75.88	78.88	83.88	90.88	99.88	110.88	123.88	138.88	$\dots$
$(1, 5)$	76.94	77.94	80.94	85.94	92.94	101.94	112.94	125.94	140.94	$\dots$
$(2, 3)$	95.28	96.28	99.28	104.28	111.28	120.28	131.28	144.28	159.28	$\dots$
$(1, 6)$	98.73	99.73	102.73	107.73	114.73	13.73	134.73	147.73	162.73	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 10.4. Approximate values of $\lambda_{k,j,l}$, where the zigzag line indicates the first 47 smallest values.

an approximate value of $z_{k,j}^2$ (cf. Table 10.3). Then, we rearrange the values $z_{k,j}^2$ in an increasing order and list approximate values of $\lambda_{k,j,l}$ accordingly (cf Table 10.4).

Choose $a = 66.5$ and $b = 69.5$. Then, by Table 10.4, one verifies that

$$\sigma(-\triangle) \cap (a,b) = \{\lambda_{1,4,3}\}.$$

Thus, the formula (10.45) reduces to

$$\begin{aligned}\omega &= \prod_{\lambda_{k,j,l}<66.5} \operatorname{Deg}_{\mathcal{V}_{j,l}} * \Big((SO(2)^{\varphi_4,3}) + (D_8^{d,3}) - (\mathbb{Z}_8^{d,3})\Big)\\ &= \prod_{(k,j,l)\in\mathcal{I}} \operatorname{Deg}_{\mathcal{V}_{j,l}} * \Big((SO(2)^{\varphi_4,3}) + (D_8^{d,3}) - (\mathbb{Z}_8^{d,3})\Big),\end{aligned}$$

where the index set $\mathcal{I}$ can be determined by the blue part of Table 10.4.

Therefore, we have

$$\begin{aligned}\omega = {}& \operatorname{Deg}_{\mathcal{V}_0} * \operatorname{Deg}_{\mathcal{V}_0} * \operatorname{Deg}_{\mathcal{V}_1} * \operatorname{Deg}_{\mathcal{V}_1} * \operatorname{Deg}_{\mathcal{V}_2} * \operatorname{Deg}_{\mathcal{V}_3} * \operatorname{Deg}_{\mathcal{V}_4}\\ & * \operatorname{Deg}_{\mathcal{V}_{0.1}} * \operatorname{Deg}_{\mathcal{V}_{0.1}} * \operatorname{Deg}_{\mathcal{V}_{1.1}} * \operatorname{Deg}_{\mathcal{V}_{1.1}} * \operatorname{Deg}_{\mathcal{V}_{2.1}} * \operatorname{Deg}_{\mathcal{V}_{3.1}} * \operatorname{Deg}_{\mathcal{V}_{4.1}}\\ & * \operatorname{Deg}_{\mathcal{V}_{0.2}} * \operatorname{Deg}_{\mathcal{V}_{0.2}} * \operatorname{Deg}_{\mathcal{V}_{1.2}} * \operatorname{Deg}_{\mathcal{V}_{1.2}} * \operatorname{Deg}_{\mathcal{V}_{2.2}} * \operatorname{Deg}_{\mathcal{V}_{3.2}} * \operatorname{Deg}_{\mathcal{V}_{4.2}}\\ & * \operatorname{Deg}_{\mathcal{V}_{0,3}} * \operatorname{Deg}_{\mathcal{V}_{0,3}} * \operatorname{Deg}_{\mathcal{V}_{1,3}} * \operatorname{Deg}_{\mathcal{V}_{1,3}} * \operatorname{Deg}_{\mathcal{V}_{2,3}} * \operatorname{Deg}_{\mathcal{V}_{3,3}}\\ & * \operatorname{Deg}_{\mathcal{V}_{0,4}} * \operatorname{Deg}_{\mathcal{V}_{0,4}} * \operatorname{Deg}_{\mathcal{V}_{1,4}} * \operatorname{Deg}_{\mathcal{V}_{1,4}} * \operatorname{Deg}_{\mathcal{V}_{2,4}} * \operatorname{Deg}_{\mathcal{V}_{3,4}}\\ & * \operatorname{Deg}_{\mathcal{V}_{0,5}} * \operatorname{Deg}_{\mathcal{V}_{0,5}} * \operatorname{Deg}_{\mathcal{V}_{1,5}} * \operatorname{Deg}_{\mathcal{V}_{2,5}} * \operatorname{Deg}_{\mathcal{V}_{3,5}} * \operatorname{Deg}_{\mathcal{V}_{0,6}}\\ & * \operatorname{Deg}_{\mathcal{V}_{0,6}} * \operatorname{Deg}_{\mathcal{V}_{1,6}} * \operatorname{Deg}_{\mathcal{V}_{2,6}} * \operatorname{Deg}_{\mathcal{V}_{0,6}} * \operatorname{Deg}_{\mathcal{V}_{1,6}}\\ & * \Big((SO(2)^{\varphi_4,3}) + (D_8^{d,3}) - (\mathbb{Z}_8^{d,3}).\Big)\end{aligned}$$

Notice that

$$\operatorname{Deg}_{\mathcal{V}_i} * (SO(2)^{\varphi_4,3}) = \begin{cases} (SO(2)^{\varphi_4,3}), & \text{if } i = 0,\\ (SO(2)^{\varphi_4,3}) - (\mathbb{Z}_i^{\varphi_4,3}), & \text{if } i = 1,2,3,4,\end{cases}$$

and for $l = 1, 2, \ldots, 7$,

$$\operatorname{Deg}_{\mathcal{V}_{j,l}} * (SO(2)^{\varphi_4,3}) = \begin{cases} (SO(2)^{\varphi_4,3}) - 2(\mathbb{Z}_4), & \text{if } j = 0,\\ (SO(2)^{\varphi_4,3}) - (\mathbb{Z}_{4-j}^{\varphi_4,l'}) - (\mathbb{Z}_{4+j}^{\varphi_j,l'}), & \text{if } j = 1,2,3,\\ (SO(2)^{\varphi_4,3}) - 2(\mathbb{Z}_8^{d,l'}), & \text{if } j = 4,\end{cases}$$

where $l' = \gcd(4, l)$. Consequently, ω contains a nontrivial $(SO(2)^{\varphi_4,3})$-term.

Similarly, we have

$$\mathrm{Deg}_{\mathcal{V}_i} * (D_8^{d,3}) = \begin{cases} (D_8^{d,3}), & \text{if } i = 0, \\ (D_8^{d,3}) - (D_1 \times \mathbb{Z}_3) - (D_1^{z,3}) + (\mathbb{Z}_1 \times \mathbb{Z}_3), & \text{if } i = 1, 3, \\ (D_8^{d,3}) - 2(D_4^{d,3}) + (\mathbb{Z}_4^{d,3}), & \text{if } i = 2, \\ -(D_8^{d,3}) + (\mathbb{Z}_8^{d,3}), & \text{if } i = 4. \end{cases}$$

Moreover, $\mathrm{Deg}_{\mathcal{V}_{j,l}} * (D_8^{d,3}) = (D_8^{d,3})$, for $0 \leq j \leq 4$ and $1 \leq l \leq 7$. Therefore, ω also contains a nontrivial $(D_8^{d,3})$-term.

Conclusion: The equation (10.36) has at least two $O(2) \times S^1$-orbits of nontrivial l-periodic solutions: one of them is a 3-folded rotating wave (or spiral vortex) and the other is a 3-folded ribbon (or stationary wave).

Part III

Appendix

A1

Sobolev Spaces and Properties of Nemitsky Operator

A1.1 Sobolev Spaces on a domain $\Omega \subset \mathbb{R}^N$

Let $\Omega \subset \mathbb{R}^N$ be an open set, $1 \le p < \infty$. We denote by $C_c^\infty(\Omega)$ the space of all smooth functions $\varphi : \Omega \to \mathbb{R}$ with compact support.

Definition A1.1.1. The *Sobolev space* $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \begin{cases} \exists_{g_1,\dots,g_N \in L^p(\Omega)} \ \forall_{\varphi \in C_c^\infty(\Omega)} \ \forall_{i=1,2,\dots,N} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi. \end{cases} \right\}$$

We put $H^1(\Omega) := W^{1,2}(\Omega)$ and will denote by $\frac{\partial u}{\partial x_i} := g_i$, $i = 1, \dots, N$, the so-called *weak derivatives* of u.

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p} := \|u\|_p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p,$$

where $\|\cdot\|_p$ is the p-norm in $L^p(\Omega)$. The space $H^1(\Omega)$ has the inner product

$$\langle u, v \rangle_{1,2} := \langle u, v \rangle_2 + \sum_{i=1}^N \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_2,$$

where $\langle \cdot, \cdot \rangle_2$ denotes the L^2-inner product in $L^2(\Omega)$, and the associated norm

$$\|u\|_{1,2} := \left[\|u\|_2^2 + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right]^{\frac{1}{2}}.$$

We have following properties of the Sobolev spaces (cf. [26, 127, 164]):

Proposition A1.1.2. *The space $W^{1,p}(\Omega)$ is a separable Banach space for $1 \le p < \infty$, which is also reflexive for $1 < p < \infty$.*

Proposition A1.1.3. (Friedrich) *Let $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$. Then there exists a sequence $\{u_n\} \in C_c^\infty(\mathbb{R}^N)$ such that*

(a)$u_n|_\Omega \to u$ in $L^p(\Omega)$;

(b)$\nabla u_n|_\omega \to \nabla u|_\omega$ in $L^p(\omega;\mathbb{R}^N)$ for every open set $\omega \Subset \Omega$ (i.e. $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$), where $\nabla u := \left[\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right]$

Proposition A1.1.4. *Let $u \in L^p(\Omega)$, $1 < p < \infty$. The following conditions are equivalent*

(i) $u \in W^{1,p}(\Omega)$;

(ii) There exists a constant C such that

$$\forall_{\varphi \in C_c^\infty(\Omega)}\, \forall_{i=1,\dots,N} \quad \left|\int_\Omega u \frac{\partial \varphi}{\partial x_i}\right| \le C\|\varphi\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(iii) There exists a constant C such that for every open sets $\omega \Subset \Omega$ we have

$$\|\tau_h u - u\|_{L^p(\omega)} \le C|h|$$

for $|h| < \operatorname{dist}(\omega, \partial\Omega)$, where $(\tau_h u)(x) := u(x+h)$.

Moreover, in the conditions (ii) and (iii) one can take C to be equal $\|\nabla u\|_p$.

A1.1.1 Sobolev Space $W^{m,p}(\Omega)$

Definition A1.1.5. The *Sobolev space* $W^{m,p}(\Omega)$, $1 \le p < \infty$, is defined for $m \ge 2$ by

$$W^{m,p}(\Omega) := \left\{ u \in W^{m-1,p}(\Omega) : \forall_{i=1,\dots,N}\ \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \right\},$$

or equivalently

$$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) : \begin{cases} \forall_\alpha\ |\alpha| \le m\ \exists_{g_\alpha \in L^p(\Omega)}\ \forall_{\varphi \in C_c^\infty(\Omega)} \\ \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi, \end{cases} \right\}$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ are multi-indices ($\alpha_j \ge 0$), $|\alpha| = \sum_{i=1}^N \alpha_i$ and $D^\alpha \varphi := \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$. We put $H^m(\Omega) := W^{m,2}(\Omega)$ and will denote by $D^\alpha u := g_\alpha$, the so-called *α-weak derivatives* of u.

A1.1.2 Embeddings of Sobolev Spaces

Definition A1.1.6. Let $\Omega \subset \mathbb{R}^N$ be an open subset. Then, Ω is called *regular* of class C^p for some $p \in [1, \infty]$, if $\partial\Omega$ is a C^p-submanifold of $\mathbb{R}^N$.

Theorem A1.1.7. (SOBOLEV EMBEDDING THEOREM) *Let $\Omega \subset \mathbb{R}^N$ be an open regular set of class C^1, where $N \geq 2$. Then,*

(i) if $p < N$ and $\frac{1}{q} + \frac{1}{p} = 1$, then for all $q' \in [1, q)$, we have the compact embedding $W^{1,p}(\Omega) \subset L^{q'}(\Omega)$;
(ii) if $p = N$, then for every $q \in [1, \infty)$, we have the compact embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$;
(iii) if $p > N$, then we have the compact embedding $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

A1.1.3 Space $W_0^{1,p}(\Omega)$

Definition A1.1.8. Let $1 \leq p < \infty$. The space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\omega)$ in $W_0^{1,p}(\Omega)$. We put $H_0^1(\Omega) := W_0^{1,2}(\Omega)$.

Proposition A1.1.9. *Assume that $\Omega \subset \mathbb{R}^N$ is an open set of class C^1. Let $u \in L^p(\Omega)$ with $1 < p < \infty$. The following properties are equivalent*

(i) $u \in W_0^{1,p}(\Omega)$;
(ii) There exists a constant c such that

$$\forall_{\varphi \in C_c^\infty(\Omega)} \forall_{i=1,\dots,N} \quad \left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} \right| \leq c\|\varphi\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Corollary A1.1.10. (POINCARÉ INEQUALITY) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $1 \leq p < \infty$. Then there exists a constant c (depending only on Ω and p) such that*

$$\forall_{u \in W_0^{1,p}(\Omega)} \quad \|u\|_p \leq c\|\nabla u\|_p.$$

In particular, $\|u\|_{W_0^{1,p}} := \|\nabla u\|_p$ is a norm in $W_0^{1,p}(\Omega)$ which is equivalent to the norm $\|u\|_{1,p}$ in $W_0^{1,p}(\Omega)$. Moreover, the expression

$$\langle u, v \rangle_{H_0^1} := \int_\Omega \nabla u \cdot \nabla v,$$

defines a scalar product on $H_0^1(\Omega)$ and the associated norm $\|u\|_{H_0^1}$ which is equivalent to the norm $\|u\|_{1,2}$.

A1.1.4 Sobolev Spaces $H^s(\Omega)$, $s \in \mathbb{R}_+$

If $u \in L^2(\mathbb{R}^n)$, the Fourier transform $\widehat{u} \in L^2(\mathbb{R}^n)$ is defined by

$$\widehat{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot y} dx, \quad y \in \mathbb{R}^n.$$

The linear operator $\mathcal{F} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $\mathcal{F}(u) := \widehat{u}$ is a symmetric isomorphism and its inverse is

$$\mathcal{F}^{-1}(v)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} v(y) e^{ix\cdot y} dy, \quad x \in \mathbb{R}^n.$$

Definition A1.1.11. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in (\mathbb{Z}_+)^n$ be multi-indices. The *Schwartz space* $\mathcal{S}$ is defined by

$$\mathcal{S} := \{u \in C^\infty(\mathbb{R}^n) : x^\alpha D^\beta u \in L^2(\mathbb{R}^n), \ \text{ for all multi-indices } \alpha, \beta\}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$. The space $\mathcal{S}$ is also called *the space of rapidly decreasing functions*.

One can easily verify the following properties of the Fourier transform $\mathcal{F}$

$$\mathcal{F}(D^\alpha u)(y) = (iy)^\alpha \mathcal{F}(u), \quad D^\beta \mathcal{F}(u)(x) = \mathcal{F}((-ix)^\beta u), \quad u \in \mathcal{S}. \tag{A1.1}$$

Using the properties (A1.1), the Sobolev space $H^m(\mathbb{R}^n)$, $m \in \mathbb{N}$, can be equivalently defined by

$$H^m(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : (1 + |y|^2)^{\frac{m}{2}} \widehat{u} \in L^2(\mathbb{R}^n)\}, \tag{A1.2}$$

equipped with the norm

$$| u |_{2,m} := \|(1 + |y|^2)^{\frac{m}{2}} \widehat{u}\|_2, \quad u \in H^m(\mathbb{R}^n).$$

Definition A1.1.12. The *fractional Sobolev spaces*, for $s > 0$, is defined as follows

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : (1 + |y|^2)^{\frac{s}{2}} \widehat{u} \in L^2(\mathbb{R}^n)\}, \tag{A1.3}$$

with the norm

$$| u |_{2,s} := \|(1 + |y|^2)^{\frac{s}{2}} \widehat{u}\|_2, \quad u \in H^s(\mathbb{R}^n).$$

The space $H^s(\mathbb{R}^n)$ is in fact a Hilbert space. Let $\Omega \subset \mathbb{R}^n$ be an open set with regular boundary. Then, the space $H^s(\Omega)$, for $s > 0$, is defined by

$$H^s(\Omega) := \{u|_\Omega : u \in H^s(\mathbb{R}^n)\}.$$

The following facts are well-known (cf. [127])

Proposition A1.1.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set with regular boundary and $s > \frac{n}{2}$. Then there exists a continuous injection*

$$H^s(\Omega) \hookrightarrow C(\overline{\Omega}).$$

Proposition A1.1.14. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with regular boundary and $s > s' > 0$. Then the injection*

$$H^s(\Omega) \hookrightarrow H^{s'}(\Omega)$$

is compact.

Consider the product space $\mathbb{R}^n \oplus \mathbb{R}^{n'}$. Denote by $\widetilde{y} = (y, y')$ the elements $y \in \mathbb{R}^n$ and $y' \in \mathbb{R}^{n'}$. Then, we can introduce

Definition A1.1.15. The *partial Sobolev space* $H^{s,s'}(\mathbb{R}^n \oplus \mathbb{R}^m)$ is defined by

$$H^{s,s'}(\mathbb{R}^n \oplus \mathbb{R}^{n'}) := \{u \in L^2(\mathbb{R}^n \oplus \mathbb{R}^{n'}) : (1+|y|^2)^{\frac{s}{2}}(1+|y'|^2)^{\frac{s'}{2}}\widehat{u} \in L^2(\mathbb{R}^n \oplus \mathbb{R}^{n'}).\}$$

For two open sets with regular boundaries $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^{n'}$, we define

$$H^{s,s'}(\Omega \times \Omega') := \{u|_{\Omega \times \Omega'} : u \in H^{s,s'}(\mathbb{R}^n \oplus \mathbb{R}^{n'})\}.$$

The space $H^{s,s'}(\Omega \times \Omega')$ is again a Hilbert space. Moreover, we have similar compact injections to those described in Proposition A1.1.14.

A1.2 Properties of The Nemitsky Operator

Definition A1.2.1. Let $\Omega \subset \mathbb{R}^N$ be an open set. A function $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ is said to satisfy the *Carathéodory conditions*, if

(i) the function $y \mapsto f(x, y)$ is continuous for a.e. $x \in \Omega$;
(ii) the function $x \mapsto f(x, y)$ is measurable for all $y \in \mathbb{R}^k$.

A function satisfying (i)—(ii) is called a *Carathéodory function.*

Definition A1.2.2. Let $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ be a Carathéodory function. Define an operator N_f on the set of functions $u : \Omega \to \mathbb{R}^k$ by

$$N_f(u)(x) = f(x, u(x)) \quad \text{for} \quad x \in \Omega,$$

and call it the *Nemitsky operator.*

If u is measurable, then $N_f(u)(x)$ is clearly measurable.

Some important properties of the Nemitsky operator are listed in the following result (cf. [110], Theorem I.2.1).

Theorem A1.2.3. *Let $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ be a Carathéodory function. If $N_f : L^p(\Omega; \mathbb{R}^k) \to L^q(\Omega; \mathbb{R}^m)$ $1 < p, q < \infty$, then N_f is continuous, takes bounded sets into bounded sets and there is a constant $c > 0$ and a function $a \in L^q(\Omega)$ such that*

$$|f(x, y)| \leq a(x) + b|y|^{p/q} \quad \textit{for a.e.} \quad x, \textit{ for all } y \in \mathbb{R}^k. \tag{A1.4}$$

Moreover, if the condition (A1.4) is satisfied, then N_f defines a continuous operator from $L^p(\Omega; \mathbb{R}^k)$ to $L^q(\Omega; \mathbb{R}^m)$.

Proposition A1.2.4. *Let $f : \overline{\Omega} \times \mathbb{R}^k \to \mathbb{R}^m$ be a Carathéodory function. Assume that for every bounded set $A \subset C(\overline{\Omega}, \mathbb{R}^k)$ there exists a function $\varphi_A \in L^p(\Omega)$, $1 \leq p < \infty$,such that for all $u \in A$ we have*

$$|f(x, u(x))| \leq \varphi_A(x) \quad \textit{a.e. } x \in \Omega \tag{A1.5}$$

Then, the Nemitsky operator $N_f : C(\overline{\Omega}; \mathbb{R}^k) \to L^p(\Omega, \mathbb{R}^m)$ is well defined, continuous and takes bounded sets into bounded sets.

Proof: First we check that $N_f(u)$ is well defined. Indeed, if $u \in C(\overline{\Omega}; \mathbb{R}^k)$, then the function $x \mapsto f(x, u(x))$ is measurable, and, by the condition (A1.5) applied to $A = \{u\}$, there is a function $\varphi_A \in L^p(\Omega)$ such that $|f(x, u(x))| \leq \varphi_A(x)$ a.e. $x \in \Omega$. Thus, $\|N_f(u)\|_p \leq \|\varphi_A\|_p < \infty$.

Now, we verify that N_f takes bounded sets into bounded sets. For, let $A \subset C(\overline{\Omega}; \mathbb{R}^k)$ be a bounded set and let $\varphi_A(x)$ be a function given by (A1.5).

Then, for every $u \in A$, we have $\|N_f(u)\|_p \leq \|\varphi_A\|_p$. Thus, $N_f(A)$ is bounded in $L^p(\Omega, \mathbb{R}^m)$.

To show that N_f is continuous, assume that $\{u_n\} \subset C(\overline{\Omega}; \mathbb{R}^k)$ is a convergent sequence to a function u. We put $A := \{u_n\}_{n=1}^{\infty} \cup \{u\}$. By (A1.5), there is a function $\varphi_A \in L^p(\Omega)$ such that $|f(x, v(x))| \leq \varphi_A(x)$ a.e. $x \in \Omega$ for all $v \in A$, thus

$$|f(x, u(x)) - f(x, u_n(x))|^p \leq 2^p |\varphi_A(x)|^p \quad \text{a.e. } x \in \Omega.$$

Since the function $f(x, \cdot)$ is continuous for a.e. x thus

$$\forall_{\varepsilon > 0} \, \exists_{N_\varepsilon} \, \forall_{n > N_\varepsilon} \; |f(x, u_n(x)) - f(x, u(x))| < \varepsilon.$$

This implies that the sequence $\{|f(x, u_n(x)) - f(x, u(x))|^p\}_{n=1}^{\infty}$ converges to zero for a.e. x. Now, by the Lebesgue's dominated convergence theorem, the sequence $\|N_f(n) - N_f(u_n)\|_1^p \to 0$ in L^1, thus $\|N_f(u) - N_f(u_n)\|_p \to 0$ as $n \to \infty$. □

In order to establish differentiability conditions for the Nemitsky operator N_f, assume that $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ is a Carethéodory function satisfying the growth condition

$$|f(x, y)| < a(x) + b|y|^p \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R}^k, \tag{A1.6}$$

where $a \in L^p(\Omega)$ and $b > 0$. Then the Nemitsky operator $N_f : L^p(\Omega; \mathbb{R}^k) \to L^1(\Omega; \mathbb{R}^m)$ is continuous. Assume that $f(x, y)$ is differentiable with respect to y and denote its derivative by $f'_y(x, y)$. Assume that $f'_y(x, y)$ is also a Carathéodory function. Then, the Nemitsky operator $N_{f'_y} : L^p(\Omega; \mathbb{R}^k) \to L^{\frac{p}{p-1}}(\Omega; \mathbb{R}^m)$ is well defined if the following growth condition is satisfied:

$$|f'_y(x, y)| \leq a_1(x) + b_1|y|^{p-1} \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R}^k, \tag{A1.7}$$

where $a_1 \in L^{\frac{p}{p-1}}(\Omega)$ and $b_1 > 0$ is a constant. Let u, $h \in L^p(\Omega; \mathbb{R}^k)$. By the Hölder Inequality,

$$\int_\Omega |f'_y(x, u(x))h(x)| dx \leq \Big[\int_\Omega |f'_y(x, u(x))|^{\frac{p}{p-1}} dx\Big]^{\frac{p-1}{p}} \Big[\int_\Omega |h(x)|^p dx\Big]^{1/p}.$$

We have the following

Proposition A1.2.5. *Assume that f satisfies the conditions (A1.4) and (A1.7). Then, the Nemitsky operator $N_f : L^p(\Omega;\mathbb{R}^k) \to L^1(\Omega;\mathbb{R}^m)$ is Fréchet C^1-differentiable and*

$$[DN_f(u)h](x) = f'_y(x,u(x))h(x), \qquad \textit{for a.e. } x \in \Omega,\ h \in L^p(\Omega;\mathbb{R}^k)$$

for all $u \in L^p(\Omega;\mathbb{R}^k)$.

Proof: Remark that for a.e. $x \in \Omega$

$$f(x,u(x)+h(x)) - f(x,u(x)) = \int_0^1 f'_y(x,u(x)+th(x))h(x)dt,$$

thus

$$\begin{aligned}
&\|N_f(u+h) - N_f(u) - N_{f'_y}(u)h\|_1\\
&= \int_\Omega |f(x,u(x)+h(x)) - f(x,u(x)) - f'_y(x,u(x))h(x)|dx\\
&= \int_\Omega \Big| \int_0^1 (f'_y(x,u(x)+th(x)) - f'_y(x,u(x)))h(x)dt\Big|dx\\
&\le \int_\Omega \int_0^1 |f'_y(x,u(x)+th(x)) - f'_y(x,u(x))|\,|Verth(x)|dtdx.
\end{aligned}$$

By Hölder inequality

$$\begin{aligned}
&\|N_f(u+h) - N_f(u) - N_{f'_y}(u)h\|_1\\
&\le \int_0^1 \left[\int_\Omega |f'_y(x,u(x)+th(x)) - f'y(x,u(x))|^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}} dt \left[\int_\Omega |h(x)|^p dx\right]^{\frac{1}{p}}.
\end{aligned}$$

Since, by Theorem A1.2.3, $N_{f'_y}$ is continuous from $L^p(\Omega;\mathbb{R}^k)$ into $L^{\frac{p}{p-1}}(\Omega;\mathbb{R}^m)$,

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{h\in L^p(\Omega;\mathbb{R}^k)} \quad \|h\|_p < \delta \ \Rightarrow\ \|N_{f'_y}(u+h) - N_{f'_y}(u+h)\|_{\frac{p}{p-1}} < \varepsilon.$$

Therefore, if $0 < \|h\|_p < \delta$,

$$\begin{aligned}
\|N_f(u+h) - N_f(u) - N_{f'_y}(u)h\|_1 &\le \int_0^1 \|N_{f'_y}(u+h) - N_{f'_y}(u)\|dt\ \|h\|_p\\
&< \varepsilon\|h\|_p.
\end{aligned}$$

This least inequality means that N_f is Fréchet differentiable at u and its derivative at u is exactly the operator $h \to N_{f'_y}(u)h$. Notice, that the operator the Nemitsky operator N_f is of class C^1. ☐

Let us point out that a more general result is true (cf. [109, 111, 112]).

Proposition A1.2.6. *Suppose that* $f : \overline{\Omega} \times \mathbb{R}^k \to \mathbb{R}^m$ *is a Caretheodory function, differentiable with respect to* y, *such that the following growth conditions are satisfied:*

$$|f(x,y)| \le a(x) + b|y|^{\frac{p}{q}} \quad \textit{for a.e. } x \in \Omega, \textit{ and all } y \in \mathbb{R}^k, \tag{A1.8}$$

where $a \in L^q(\Omega)$;

$$|f'_y(x,y)| \le a_1(x) + b_1|y|^{\frac{p-q}{q}} \quad \textit{for a.e. } x \in \Omega \textit{ and all } y \in \mathbb{R}^k,, \tag{A1.9}$$

where $a_1 \in L^{\frac{pq}{p-q}}(\Omega)$ *and* $p > q \ge 1$. *Then* $N_f : L^p(\Omega;\mathbb{R}^k) \to L^q(\Omega;\mathbb{R}^m)$ *is Fréchet* C^1*-differentiable and* $[DN_f(u)]h = N_{f'_y}(u)h$.

Assume for simplicity that $k = m = 1$. Then the Nemitsky operator $N_f : L^2(\Omega) \to L^2(\Omega)$ is continuous if and only if

$$|f(x,y)| \le a(x) + b|y| \quad \text{for a.e. } x \in \Omega \quad \text{and all } y \in \mathbb{R}.$$

On the other hand, in order to assure that N_f is Fréchet C^1-differentiable, the condition (A1.9) implies that $f(x,y) = \alpha(x) + \beta \cdot y$ for some $\alpha \in L^2(\Omega)$ and a constant $\beta > 0$. Therefore, there is no nonlinear with respect to y Carathéodory functions $f(x,y)$ such that $N_f : L^2(\Omega) \to L^2(\Omega)$ is Fréchet C^1-differentiable. In order to overcome this difficulty, assume that if there is a constant $M > 0$ such that

$$|f'_y(x,y)| \le M \quad \text{for a.e. } x \in \Omega \text{ and all } y \in \mathbb{R}.$$

Then, N_f is Gâteaux differentiable on $L^2(\Omega)$.

Proposition A1.2.7. *Let* $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ *be a Carathéodory function, differentiable with respect to* y *such that* $f'_y(x,y)$ *is also a Carathéodory function. Suppose that the following conditions are satisfied:*

(i) *there is a function* $a \in L^2(\Omega)$ *and a constant* $b > 0$ *such that*

$$|f(x,y)| \le a(x) + b|y| \quad \textit{for a.e. } x \in \Omega \textit{ and all } y \in \mathbb{R}^k;$$

(ii) *there is a constant* $M > 0$ *such that*

$$|f'_y(x,y)| \le M \quad \textit{for a.e. } x \in \Omega \textit{ and all } y \in \mathbb{R}^k.$$

Then, the Nemitsky operator $N_f : L^2(\Omega;\mathbb{R}^k) \to L^2(\Omega;\mathbb{R}^m)$ *is Gâteaux differentiable and*

$$[DN_f(u)h](x) = f'_y(x,u(x))h(x).$$

Proof: Let u, $h \in L^2(\Omega;\mathbb{R}^k)$. We have

$$\left[\int_\Omega \left|\frac{1}{t}(f(x,u(x)+th(x)) - f(x,u(x))) - f'_y(x,u(x))h(x)\right|^2 dx\right]^{\frac{1}{2}}$$

$$= \left[\int_\Omega \left|\frac{1}{t}\int_0^1 f'_y(x,u(x)+sth(x))th(x)ds - f'_y(x,u(x))h(x)\right|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \left[\int_\Omega \left[\int_0^1 (|f'_y(x,u(x)+sth(x)) - f'_y(x,u(x))h(x)|ds\right]^2 dx\right]^{\frac{1}{2}}.$$

Notice that $\underset{t\to 0}{\to}\lim |f'_y(x,u(x)+sth(x)) - f'_y(x,u(x))| = 0$ for a.e. x, thus by (ii),

$$\left[\int_0^1 (|f'_y(x,u(x)+sth(x)) - f'_y(x,u(x))h(x)|ds\right]^2 \leq 4M^2 \int_0^1 |h(x)|^2 dx < \infty$$

and by the Lebesgue's Dominated Convergence Theorem,

$$\lim_{t\to 0}\left[\int_\Omega \left|\frac{1}{t}(f(x,u(x)+th(x)) - f(x,u(x))) - f'_y(x,u(x))h(x)\right|^2 dx\right]^{1/2} = 0.$$

Consequently, Gâteaux derivative of N_f at u is the operator $h \to N_{f'_y}(u)h$. □

A1.3 Differentiability of Functionals on Sobolev Space $H^1(\Omega)$

Assume that $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an open bounded regular of class C^1 set, $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ is a twice differentiable function with respect to y such that $f(x,y)$, $f'_y(x,y)$ and $f''_y(x,y)$ are Carathéodory functions for which the following conditions are satisfied

$$|f(x,y)| \leq a(x) + b|y|^{\frac{p}{q}} \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R}^k \tag{A1.10}$$

$$|f'_y(x,y)| \leq c(x) + d|y|^{\frac{p-q}{q}} \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R}^k \tag{A1.11}$$

$$|f''_y(x,y)| \leq e(x) + g|y|^{\frac{p-2q}{q}} \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R}^k \tag{A1.12}$$

where $p > 2q \geq 2$, $a \in L^q(\Omega)$, $c \in L^{\frac{pq}{p-q}}(\Omega)$, $e \in L^{\frac{pq}{p-2q}}(\Omega)$, and b, d, $g > 0$ are constants.

We have the following

Corollary A1.3.1. *Under the conditions (A1.10)—(A1.12), the Nemitsky operator* $N_f : L^p(\Omega; \mathbb{R}^k) \to L^q(\Omega, \mathbb{R}^m)$ *is twice differentiable of class* C^2 *and*

$$D^2 N_f(\varphi)(h, g)(x) = h(x) f''_y(x, \varphi(x)) g(x); \quad \varphi, h, g \in L^p(\Omega).$$

For simplicity, assume that $m = k = 1$. The same results hold for more general case. The inclusion $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is well defined, continuous and compact whenever (cf. Theorem A1.1.7)

$$p < \frac{2N}{N-2}.$$

By Theorem A1.2.3, the operator $N_f : L^p(\Omega) \to L^1(\Omega)$ is well defined if f satisfies the Carathéodory conditions and

$$|f(x,y)| \le a(x) + b|y|^p \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R},$$

where $a \in L^1(\Omega)$.

Consider a functional $\Psi : H^1(\Omega) \to \mathbb{R}$ defined as the following composition

$$H^1(\Omega) \hookrightarrow L^p(\Omega) \xrightarrow{N_f} L^1(\Omega) \xrightarrow{\langle 1, \cdot \rangle} \mathbb{R}$$

where $\langle 1, u \rangle := \int_\Omega u(x) dx$, and f is a function. Clearly, the functional $\Psi : H_1(\Omega) \to \mathbb{R}$ is given by

$$\Psi(u) = \int_\Omega f(x, u(x)) dx, \quad u \in H^1(\Omega).$$

Assume that the function f is twice differentiable with respect to the variable y and that f, f'_y, f''_y are Carathéodory functions and that the following conditions are satisfied

$$|f(x,y)| \le a(x) + c|y|^p \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R} \tag{A1.13}$$
$$|f'_y(x,y)| \le b(x) + d|y|^{p-1} \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R} \tag{A1.14}$$

where $a \in L^1(\Omega)$, $b \in L^{\frac{p}{p-1}}(\Omega)$, c, $d > 0$. The condition (A1.14) implies that N_f is Fréchet differentiable of class C^1 and that

$$[DN_f(u)h](x) = (N_{f'_y}(u) \cdot h)(x) = f'_y(x, u(x)) h(x).$$

Therefore, Ψ is a differentiable functional of class C^1 and

$$D\Psi(u)h = \int_\Omega f'_y(x,u(x))h(x)dx.$$

The condition (A1.14) can be rewritten as follows

$$|f'_y(x,y)| \le b(x) + c|y|^\beta \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R}, \tag{A1.15}$$

where $b \in L^{\frac{\beta+1}{\beta}}(\Omega)$ and $\beta \le \frac{N+2}{N-2}$.

Similarly, in order to have that Ψ is of class C^2 we need assure differentiability of $N_{f'_y}$, for which we need that f''_y satisfies the Carathéodory conditions and

$$|f''_y(x,y)| \le e(x) + g|y|^\gamma \quad \text{for a.e. } x \in \Omega \text{ and for all } y \in \mathbb{R},$$

where $e \in L^{\frac{\gamma+2}{\gamma}}(\Omega)$ and $\gamma \le \dfrac{4}{N-2}$ (i.e. $p-2=\gamma$ and $p \le \frac{2N}{N-2}$).

Now, we define the functional $J : H^1_0(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \Psi(u), \quad u \in H^1_0(\Omega)$$

where $\Psi(u) = \int_\Omega f(x,u(x))dx$, f and f'_y satisfy the Carathéodory conditions and the conditions (A1.13) and (A1.15), with $\beta < \frac{N+2}{N-2}$. Consequently, J is of class C^1 and

$$\begin{aligned} DJ(u)h &= \int_\Omega \nabla u \nabla h - \int_\Omega f'_y(x,u(x)) \cdot h(x)dx \\ &= \int_\Omega \nabla u \nabla h - D\Psi(u)h. \end{aligned}$$

Let $\tau : L^{\frac{p}{p-1}}(\Omega) \to (H^1_0(\Omega))^* =: H^{-1}(\Omega)$ be defined by $\tau(h)(u) = \int_\Omega hu$, where $h \in L^{\frac{p}{p-1}}(\Omega)$, $u \in H^1_0(\Omega)$. The operator τ is well defined and continuous.

Indeed, if $p \ge 2$, then by applying the Hölder inequality, the Poincaré inequality and Theorem A1.1.7, we obtain

$$\begin{aligned} \left|\int_\Omega hu\right| &\le \|h\|_{p/(p-1)} \cdot \|u\|_p \le \tilde{c}\|h\|_{p/(p-1)} \cdot \|u\|_{p,1} \\ &\le c\|h\|_{p/(p-1)} \cdot \|u\|_{H^1_0}. \end{aligned}$$

If $p < 2$, then $\frac{p}{p-1} > 2$ and $L^{\frac{p}{p-1}}(\Omega) \subset L^2(\Omega)$, thus

$$\left|\int_\Omega hu\right| \le \tilde{c}\|h\|_2\|u\|_{H_0^1}.$$

Let $R : L^{p/p-1}(\Omega) \to H_0^1(\Omega)$ be the composition of τ with the isomorphism $(H_0^1(\Omega))^* \cong H_0^1(\Omega)$ given by Riesz theorem. This means that Rh is the unique solution to the problem

$$\forall_{\varphi\in H_0^1(\Omega)} \quad \langle \nabla u, \nabla\varphi\rangle_{L^2} = \int_\Omega h\varphi$$

i.e. Rh is a weak solution to the problem

$$-\Delta u = h, \quad u|_{\partial\Omega} \equiv 0.$$

Using the operator R, we can calculate $\nabla\Psi(u)$. Since

$$D\Psi(u)h = \int_\Omega f_y'(x, u(x))h(x)dx,$$

where $N_{f_y'}(u) \in L^{\frac{p}{p-1}}(\Omega)$, we have

$$\nabla\Psi(u) = RN_{f_y'}(u)$$

which means the $\nabla\Psi$ is the following composition

$$\begin{tikzcd}
H_0^1(\Omega) \arrow[r, "\nabla\Psi"] \arrow[d, "i"'] & H_0^1(\Omega) \\
L^p(\Omega) \arrow[r, "N_{f_y'}"] & L^{\frac{p}{p-1}}(\Omega) \arrow[u, "R"']
\end{tikzcd}$$

Consequently, we have the following result

Proposition A1.3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded regular of class C^1 set. If $\beta = p - 1 < \frac{N+2}{N-2}$, then $\nabla\Psi : H_0^1(\Omega) \to H_0^1(\Omega)$ is a completely continuous operator.*

Proof: By Theorem A1.1.7, the inclusion $i : H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is a compact operator, thus $\nabla\Psi$ is completely continuous. □

A2

Catalogue of Groups

A2.1 Groups and Their Subgroups

In this section, we classify and catalog a list of the subgroups in Γ and $\Gamma \times S^1$, up to their conjugacy classes, where Γ takes values of the quaternionic units group Q_8, the dihedral group D_N, the tetrahedral group A_4, the octahedral group S_4, the icosahedral group A_5, the orthogonal group $O(2)$ and the tori group T^N.

There are two types of subgroups in $\Gamma \times S^1$,

(i) $K \times S^1$, for a subgroup $K \subset \Gamma$;
(ii) the φ-twisted l-folded subgroups $K^{\varphi,l}$, for a homomorphism $\varphi : K \to S^1$ and $l \in \{0\} \cup \mathbb{N}$ (cf. Subsection 4.2.1),

where in (ii), notice that $K^{\varphi,0} = K \times \{1\}$, and $K^{\varphi,l}$ (for $l > 1$) can be easily obtained from K^φ by

$$K^{\varphi,l} = \{(\gamma, z) \in K \times S^1 \ : \ (\gamma, z^l) \in K^\varphi\}.$$

Therefore, in what follows, we only provide a catalogue of the subgroups in Γ and the twisted one-folded subgroups in $\Gamma \times S^1$, up to their conjugacy classes.

A2.1.1 Quaternionic Units Group Q_8

Denote by $\mathbb{H} := \{z_1 + jz_2; \ \ z_1, \ z_2 \in \mathbb{C}\}$ the algebra of quaternions, with the multiplication rules $i^2 = j^2 = -1$, $ji = -ij$. Define

$$Q_8 := \{\pm 1, \pm i, \pm j, \pm ji\} \subset \mathbb{H}$$

to be the *quaternionic units group*. There are six subgroups in Q_8, namely

$$\mathbb{Z}_1 = \{1\}, \ \mathbb{Z}_2 = \{1, -1\}, \ \mathbb{Z}_4^1 = \{1, -1, i, -i\},$$
$$\mathbb{Z}_4^2 = \{1, -1, j, -j\}, \ \mathbb{Z}_4^3 = \{1, -1, ij, -ij\}, \ Q_8 = \{\pm 1, \pm i, \pm j, \pm ji\},$$

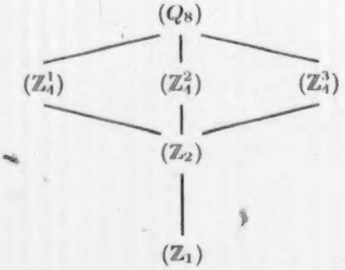
(Q_8)
$(\mathbb{Z}_4^1)$
$(\mathbb{Z}_4^2)$
$(\mathbb{Z}_4^3)$
$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$

Fig. A2.1. The lattice of conjugacy classes of subgroups in Q_8.

which represent distinct conjugacy classes. The lattice of their conjugacy classes is shown in Figure A2.1.

There are ten twisted one-folded subgroups in $Q_8 \times S^1$, namely

$$\begin{aligned}
\mathbb{Z}_2^- &= \{(1,1),(-1,-1)\},\\
\mathbb{Z}_4^{1-} &= \{(1,1),(i,-1),(-1,1),(-i,-1)\},\\
\mathbb{Z}_4^{2-} &= \{(1,1),(j,-1),(-1,1),(-j,-1)\},\\
\mathbb{Z}_4^{3-} &= \{(1,1),(ij,-1),(-1,1),(-ij,-1)\},\\
\mathbb{Z}_4^{1+} &= \{(1,1),(i,i),(-1,-1),(-i,-i)\},\\
\mathbb{Z}_4^{2+} &= \{(1,1),(j,i),(-1,-1),(-j,-i)\},\\
\mathbb{Z}_4^{3+} &= \{(1,1),(ij,i),(-1,-1),(-ij,-i)\},\\
Q_8^{1-} &= \{(1,1),(i,1),(-1,1),(-i,1),(j,-1),(ji,-1),(-j,-1),(-ji,-1)\},\\
Q_8^{2-} &= \{(1,1),(i,-1),(-1,1),(-i,-1),(j,1),(ji,-1),(-j,1),(-ji,-1)\},\\
Q_8^{3-} &= \{(1,1),(i,-1),(-1,1),(-i,-1),(j,-1),(ji,1),(-j,-1),(-ji,1)\},
\end{aligned}$$

The lattice of the conjugacy classes of the twisted subgroups is shown in Figure A2.2.

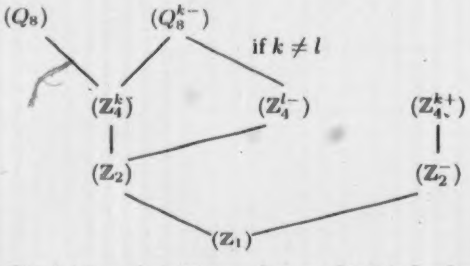
(Q_8)
(Q_8^{k-})
if $k \neq l$
$(\mathbb{Z}_4^k)$
$(\mathbb{Z}_4^{l-})$
$(\mathbb{Z}_4^{k+})$
$(\mathbb{Z}_2)$
$(\mathbb{Z}_2^-)$
$(\mathbb{Z}_1)$

Fig. A2.2. The lattice of conjugacy classes of twisted subgroups in $Q_8 \times S^1$.

A2.1.2 Dihedral Group D_N

Represent the *dihedral group* D_N of order $2N$ as the group of rotations 1, ξ, $\xi^2, \dots, \xi^{N-1}$ of the complex plane (where ξ is the multiplication by $e^{\frac{2\pi i}{N}}$) plus the reflections κ, $\kappa\xi$, $\kappa\xi^2, \dots, \kappa\xi^{N-1}$ with κ being the operator of complex conjugation described by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

For an integer $k|N$ and $\gamma := e^{i\frac{2\pi}{k}}$, the dihedral group D_N has the subgroups

$$\begin{aligned}
\mathbb{Z}_k &= \left\{1, \gamma, \gamma^2, \dots, \gamma^{k-1}\right\},\\
D_k &= \left\{1, \gamma, \gamma^2, \dots, \gamma^{k-1}, \kappa, \kappa\gamma, \dots, \kappa\gamma^{k-1}\right\},\\
D_{k,j} &= \left\{1, \gamma, \gamma^2, \dots, \gamma^{k-1}, \kappa\xi^j, \kappa\xi^j\gamma, \dots, \kappa\xi^j\gamma^{k-1}\right\},
\end{aligned}$$

where $j = 1, \dots, \frac{N}{k}-1$. The subgroup $\mathbb{Z}_k$ is normal in D_N. While the subgroups $D_{k,j}$ for $j = 0, 1, \dots, \frac{N}{k} - 1$, are all conjugate if $\frac{N}{k}$ is odd, but split into two conjugacy classes (D_k) and $(\widetilde{D}_k)$, where $\widetilde{D}_k := D_{k,1}$, if $\frac{N}{k}$ is even.

The twisted subgroups of $D_N \times S^1$ are listed as follows, for $k|N$,

$$\begin{aligned}
\mathbb{Z}_k^{t_r} &= \left\{(1,1), (\gamma, \gamma^r), (\gamma^2, \gamma^{2r}), \dots, (\gamma^{k-1}, \gamma^{(k-1)r})\right\},\\
D_k^z &= \left\{(1,1), (\gamma, 1), \dots, (\gamma^{k-1}, 1), (\kappa, -1), (\kappa\gamma, -1), \dots, (\kappa\gamma^{k-1}, -1)\right\},\\
D_{k,j}^z &= \left\{(1,1), (\gamma, 1), \dots, (\gamma^{k-1}, 1), (\kappa\xi^j, -1), (\kappa\xi^j\gamma, -1), \dots, (\kappa\xi^j\gamma^{k-1}, -1)\right\},
\end{aligned}$$

where $r \in \{1, \dots, k-1\}$ and $j = 1, \dots, \frac{N}{k} - 1$. For $0 < r < \frac{k}{2}$, $\kappa\mathbb{Z}_k^{t_r}\kappa = \mathbb{Z}_k^{t_{k-r}}$, i.e. $\mathbb{Z}_k^{t_r}$ and $\mathbb{Z}_k^{t_{k-r}}$ are conjugate. The conjugacy relations among $D_{k,j}^z$ are similar to $D_{k,j}$, for $j = 0, 1, \dots, \frac{N}{k} - 1$.

In the case $k = 2m$, we have additional twisted subgroups

$$\begin{aligned}
\mathbb{Z}_{2m}^d &= \{(1,1), (\gamma, -1), \dots, (\gamma^{2m-1}, -1)\},\\
D_{2m}^d &= \left\{(1,1), (\gamma, -1), \dots, (\gamma^{k-1}, -1), (\kappa, 1), (\kappa\gamma, -1), \dots, (\kappa\gamma^{k-1}, -1)\right\},\\
\widetilde{D}_{2m}^d &= \{(1,1), (\gamma, -1), \dots, (\gamma^{k-1}, -1), (\kappa\xi, 1), (\kappa\xi\gamma, -1), \dots, (\kappa\xi\gamma^{k-1}, -1)\},\\
\hat{D}_{2m}^{\hat{d}} &= \{(1,1), (\gamma, -1), \dots, (\gamma^{k-1}, -1), (\kappa, -1), (\kappa\gamma, 1), \dots, (\kappa\gamma^{k-1}, 1)\},
\end{aligned}$$

where $\mathbb{Z}_k^{t_r}$ is a normal subgroup, D_{2m}^d is conjugate to $\widetilde{D}_{2m}^d$ iff $\frac{N}{2m}$ is odd, while D_{2m}^d and $D_{2m}^{\hat{d}}$ are conjugate iff $\frac{N}{2m}$ is even.

Example A2.1.1. As an example, we provide a list of subgroups in D_6, and the twisted subgroups in $D_6 \times S^1$ (cf. [15]). Put $\mu := e^{i\frac{2\pi}{6}}$, then we have the following subgroups in D_6.

$$\begin{aligned}
&\mathbb{Z}_1 = \{1\}, \quad \mathbb{Z}_2 = \{1, -1\}, \quad \mathbb{Z}_3 = \{1, \mu^2, \mu^4\},\\
&\mathbb{Z}_6 = \{1, \mu, \mu^2, \mu^3, \mu^4, \mu^5\}, \quad D_1 = \{1, \kappa\}, \quad \widetilde{D}_1 = \{1, \kappa\mu\},\\
&D_2 = \{1, -1, \kappa, -\kappa\}, \quad D_3 = \{1, \mu^2, \mu^4, \kappa, \kappa\mu^2, \kappa\mu^4\},\\
&\widetilde{D}_3 = \{1, \mu^2, \mu^4, \kappa\mu, \kappa\mu^3, \kappa\mu^5\},\\
&D_6 = \{1, \mu, \mu^2, \mu^3, \mu^4, \mu^5, \kappa, \kappa\mu, \kappa\mu^2, \kappa\mu^3, \kappa\mu^4, \kappa\mu^5\}.
\end{aligned}$$

The twisted subgroups of $D_6 \times S^1$ are listed below.

$$\begin{aligned}
\mathbb{Z}_2^- &= \{(1,1), (-1,-1)\}, \quad \mathbb{Z}_3^t = \{(1,1), (\mu^2, \mu^2), (\mu^4, \mu^4)\},\\
\mathbb{Z}_6^{t_1} &= \{(1,1), (\mu, \mu), (\mu^2, \mu^2), (\mu^3, \mu^3), (\mu^4, \mu^4), (\mu^5, \mu^5)\},\\
\mathbb{Z}_6^{t_2} &= \{(1,1), (\mu, \mu^2), (\mu^2, \mu^4), (\mu^3, 1), (\mu^4, \mu^2), (\mu^5, \mu^4)\},\\
\mathbb{Z}_6^d &= \{(1,1), (\mu, -1), (\mu^2, 1), (\mu^3, -1), (\mu^4, 1), (\mu^5, -1)\},\\
D_1^z &= \{(1,1), (\kappa, -1)\}, \quad \widetilde{D}_1^z = \{(1,1), (\kappa\mu, -1)\},\\
D_2^z &= \{(1,1), (-1,1), (\kappa, -1), (-\kappa, -1)\},\\
D_2^d &= \{(1,1), (-1,-1), (\kappa, 1), (-\kappa, -1)\},\\
D_2^{\hat{d}} &= \{(1,1), (-1,-1), (\kappa, -1), (-\kappa, 1)\},\\
D_3^z &= \{(1,1), (\mu^2, 1), (\mu^4, 1), (\kappa, -1), (\kappa\mu^2, -1), (\kappa\mu^4, -1)\},\\
\widetilde{D}_3^z &= \{(1,1), (\mu^2, 1), (\mu^4, 1), (\kappa\mu, -1), (\kappa\mu^3, -1), (\kappa\mu^5, -1)\},\\
D_6^z &= \{(1,1), (\mu, 1), (\mu^2, 1), (\mu^3, 1), (\mu^4, 1), (\mu^5, 1), (\kappa, -1),\\
&\qquad (\kappa\mu, -1), (\kappa\mu^2, -1), (\kappa\mu^3, -1), (\kappa\mu^4, -1), (\kappa\mu^5, -1)\},\\
D_6^d &= \{(1,1), (\mu, -1), (\mu^2, 1), (\mu^3, -1), (\mu^4, 1), (\mu^5, -1), (\kappa, 1),\\
&\qquad (\kappa\mu, -1), (\kappa\mu^2, 1), (\kappa\mu^3, -1), (\kappa\mu^4, 1), (\kappa\mu^5, -1)\},\\
D_6^{\hat{d}} &= \{(1,1), (\mu, -1), (\mu^2, 1), (\mu^3, -1), (\mu^4, 1), (\mu^5, -1), (\kappa, -1),\\
&\qquad (\kappa\mu, 1), (\kappa\mu^2, -1), (\kappa\mu^3, 1), (\kappa\mu^4, -1), (\kappa\mu^5, 1)\}.
\end{aligned}$$

The lattice of conjugacy classes of the twisted subgroups in $D_6 \times S^1$ is illustrated in Figure A2.3.

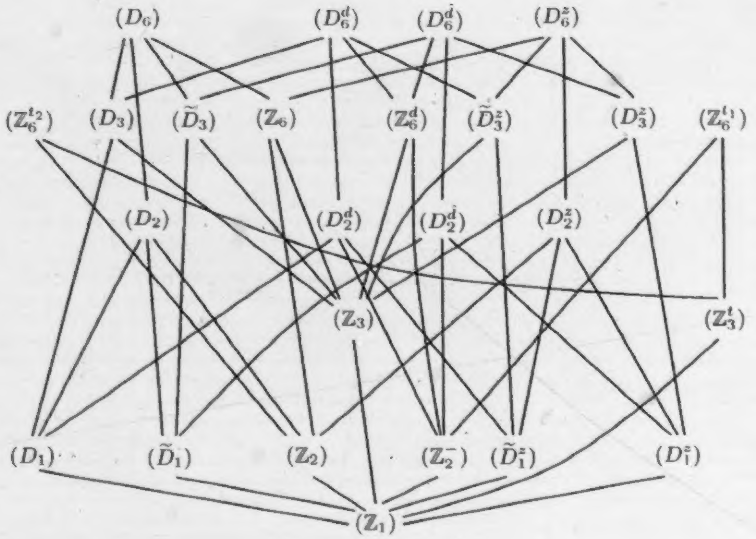
(D_6)
(D_6^d)
$(D_6^{\hat{d}})$
(D_6^z)
$(\mathbb{Z}_6^{t_2})$
(D_3)
$(\widetilde{D}_3)$
$(\mathbb{Z}_6)$
$(\mathbb{Z}_6^d)$
$(\widetilde{D}_3^z)$
(D_3^z)
$(\mathbb{Z}_6^{t_1})$
(D_2)
(D_2^d)
$(D_2^{\hat{d}})$
(D_2^z)
$(\mathbb{Z}_3)$
$(\mathbb{Z}_3^t)$
(D_1)
$(\widetilde{D}_1)$
$(\mathbb{Z}_2)$
$(\mathbb{Z}_2^-)$
$(\widetilde{D}_1^z)$
(D_1^z)
$(\mathbb{Z}_1)$

Fig. A2.3. Lattice of conjugacy classes of twisted subgroups in $D_6 \times S^1$.

A2.1.3 Tetrahedral Group A_4

It is well known that there are only five regular polyhedra: the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron. The groups of motions of regular polyhedra are called *regular polyhedral groups*. Two regular polyhedra are called *dual* to each other, if one can be obtained from the other by taking as vertices the centers of all the faces of the other polyhedron. The hexahedron and octahedron are dual to each other, as are the dodecahedron and icosahedron. The tetrahedron is dual to itself. Accordingly, the groups of motions of dually corresponding regular polyhedra are isomorphic. Hence, we speak of the tetrahedral group A_4, the octahedral group S_4 and the icosahedral group A_5.

Consider the *tetrahedral group* A_4, which consists of even permutations of four symbols $\{1, 2, 3, 4\}$. We have the following subgroups in A_4, up to their conjugacy classes

$$
\begin{aligned}
&\mathbb{Z}_1 = \{(1)\}, \quad \mathbb{Z}_2 = \{(1), (12)(34)\}, \quad \mathbb{Z}_3 = \{(1), (123), (132)\},\\
&V_4 = \{(1), (12)(34), (13)(24), (14)(23)\},\\
&A_4 = \{(1), (12)(34), (123), (132), (13)(24), (142),\\
&\qquad (124), (14)(23), (134), (143), (243), (234)\}.
\end{aligned}
$$

The lattice of the conjugacy classes of the subgroups in A_4 is shown in Figure A2.4.

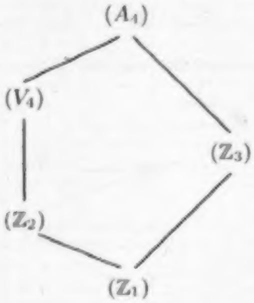

Fig. A2.4. Lattice of conjugacy classes of subgroups in A_4

The twisted subgroups in $A_4 \times S^1$ are listed as follows

$$
\begin{aligned}
\mathbb{Z}_2^- &= \{((1),1), ((12)(34),-1)\},\\
\mathbb{Z}_3^{t_1} &= \{((1),1), ((123),\gamma), ((132),\gamma^2)\},\\
\mathbb{Z}_3^{t_2} &= \{((1),1), ((123),\gamma^2), ((132),\gamma)\},\\
V_4^- &= \{((1),1), ((12)(34),1), ((13)(24),-1), ((14)(23),-1)\},\\
A_4^{t_1} &= \{((1),1), ((12)(34),1), ((13)(24),1), ((14)(23),1), ((123),\gamma),\\
&\quad ((132),\gamma^2), ((142),\gamma), ((124),\gamma^2), ((134),\gamma), ((143),\gamma^2),\\
&\quad ((243),\gamma), ((234),\gamma^2)\},\\
A_4^{t_2} &= \{((1),1), ((12)(34),1), ((13)(24),1), ((14)(23),1), ((123),\gamma^2),\\
&\quad ((132),\gamma), ((142),\gamma^2), ((124),\gamma), ((134),\gamma^2), ((143),\gamma),\\
&\quad ((243),\gamma^2), ((234),\gamma)\},
\end{aligned}
$$

where $\gamma = e^{i\frac{2\pi}{3}}$. The lattice of the conjugacy classes of subgroups in $A_4 \times S^1$ is shown on Figure A2.5.

A2.1.4 Octahedral Group S_4

Consider the *octahedral group* S_4, which consists of permutations of four symbols $\{1, 2, 3, 4\}$. Since A_4 is a subgroup of S_4, it is clear that all the subgroups

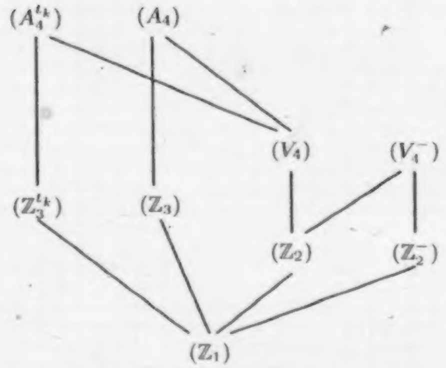

Fig. A2.5. Lattice of conjugacy classes of twisted subgroups in $A_4 \times S^1$

of A_4, namely A_4, V_4, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$, are also subgroups of S_4 (cf. Subsection A2.1.3). In addition, there are the following subgroups in S_4, up to their conjugacy classes

$$D_1 = \{(1), (12)\},$$
$$D_2 = \{(1), (12)(34), (12), (34)\},$$
$$D_3 = \{(1), (123), (132), (12), (23), (13)\},$$
$$\mathbb{Z}_4 = \{(1), (1324), (12)(34), (1423)\},$$
$$D_4 = \{(1), (1324), (12)(34), (1423), (34), (14)(23), (12), (13)(24)\}.$$

The twisted subgroups of $A_4 \times S^1$ as listed in Subsection A2.1.3, represent four conjugacy classes of twisted subgroups in $S_4 \times S^1$, namely $(\mathbb{Z}_2^-)$, $(\mathbb{Z}_3^t) := (\mathbb{Z}_3^{t_k})$ (for $k = 1, 2$), (V_4^-), and $(A_4^t) := (A_4^{t_k})$ (for $k = 1, 2$). Besides, we have additional twisted subgroups in $S_4 \times S^1$, namely

$$D_1^z = \{((1), 1), ((12), -1)\},$$
$$D_2^z = \{((1), 1), ((12)(34), 1), ((12), -1), ((34), -1)\},$$
$$D_2^d = \{((1), 1), ((12)(34), -1), ((12), 1), ((34), -1)\},$$
$$\mathbb{Z}_4^c = \{((1), 1), ((1324), i), ((12)(34), -1), ((1423), -i)\},$$
$$\mathbb{Z}_4^- = \{((1), 1), ((1324), -1), ((12)(34), 1), ((1423), -1)\},$$
$$D_3^z = \{((1), 1), ((123), 1), ((132), 1), ((12), -1), ((23), -1), ((13), -1)\},$$
$$D_4^d = \{((1), 1), ((1324), -1), ((12)(34), 1), ((1423), -1), ((34), 1), ((14)(23), -1), ((12), 1), ((13)(24), -1)\},$$

$$
\begin{aligned}
D_4^{\hat{d}} = \{&((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),-1),\\
&((14)(23),1),((12),-1),((13)(24),1)\},\\
D_4^{z} = \{&((1),1),((1324),1),((12)(34),1),((1423),1),((34),-1),\\
&((14)(23),-1),((12),-1),((13)(24),-1)\},\\
S_4^{-} = \{&((1),1),((12),-1),((12)(34),1),((123),1),((1234),-1),((13),-1),\\
&((13)(24),1),((132),1),((1342),-1),((14),-1),((14)(23),1),((142),1),\\
&((1324),-1),((23),-1),((124),1),((1243),-1),((24),-1),((134),1),\\
&((1423),-1),((34),-1),((143),1),((1432),-1),((243),1),((234),1)\}.
\end{aligned}
$$

The lattice of the conjugacy classes of subgroups in S_4 is shown in Figure A2.6 and the lattice of the conjugacy classes of the twisted subgroups in $S_4 \times S^1$ is shown on Figure A2.7.

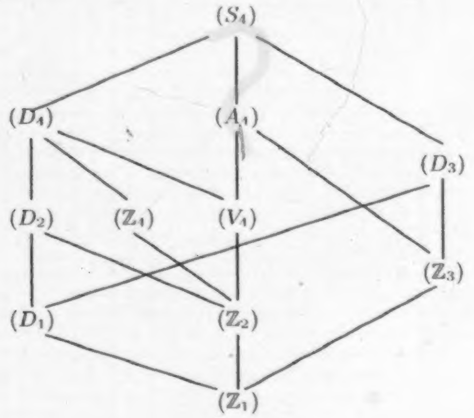

Fig. A2.6. Lattice of conjugacy classes in S_4

A2.1.5 Icosahedral Group A_5

Consider the *icosahedral group*, which consists of even permutations of five symbols $\{1,2,3,4,5\}$. Besides A_5 and $\mathbb{Z}_1$, there are seven subgroups in A_5, namely

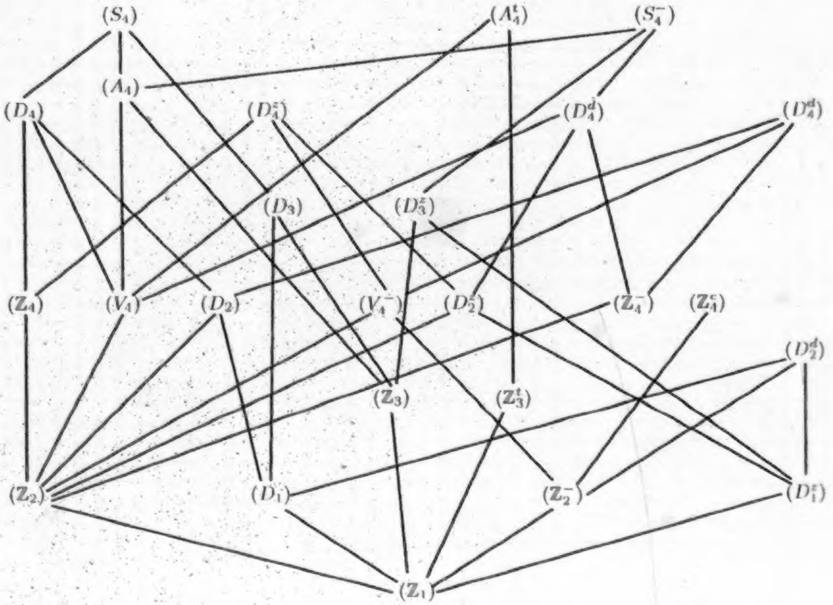
(S_4)
(A_4^t)
(S_4^-)
(A_4)
(D_4)
(D_4^z)
(D_4^d)
(D_4^d)
(D_3)
(D_3^z)
$(\mathbb{Z}_4)$
(V_4)
(D_2)
(V_4^-)
(D_2^z)
$(\mathbb{Z}_4^-)$
$(\mathbb{Z}_4^c)$
(D_2^d)
$(\mathbb{Z}_3)$
$(\mathbb{Z}_3^t)$
$(\mathbb{Z}_2)$
(D_1)
$(\mathbb{Z}_2^-)$
(D_1^z)
$(\mathbb{Z}_1)$

Fig. A2.7. Lattice of conjugacy classes of twisted subgroups in $S_4 \times S^1$.

$\mathbb{Z}_2 = \{(1), (12)(34)\}$,
$\mathbb{Z}_3 = \{(1), (123), (132)\}$,
$V_4 = \{(1), (12)(34), (13)(24), (23)(14)\}$,
$\mathbb{Z}_5 = \{(1), (12345), (13524), (14253), (15432)\}$,
$D_3 = \{(1), (123), (132), (12)(45), (13)(45), (23)(45)\}$,
$A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$,
$D_5 = \{(1), (12345), (13524), (14253), (15432), (12)(35), (13)(45), (14)(23), (15)(24), (25)(34)\}$.

The lattice of the conjugacy classes of the subgroups in A_5 is shown in Figure A2.8. The twisted subgroups in $A_5 \times S^1$ are listed as follows.

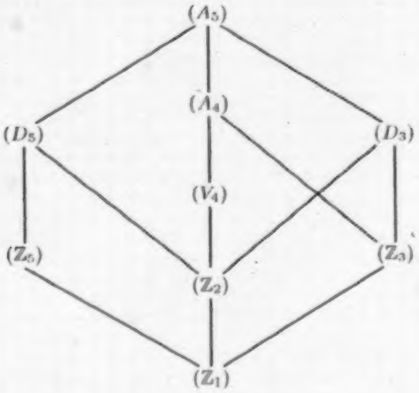

Fig. A2.8. Lattice of conjugacy classes for A_5

$$\begin{aligned}
\mathbb{Z}_2^- &= \Big\{((1),1),\ ((12)(34),-1)\Big\},\\
V_4^- &= \Big\{((1),1),\ ((12)(34),-1),\ ((13)(24),-1),\ ((23)(14),1)\Big\},\\
\mathbb{Z}_5^{t_1} &= \Big\{((1),1),\ ((12345),\xi),\ ((13524),\xi^2),\ ((14253),\xi^3),\ ((15432),\xi^4)\Big\},\\
\mathbb{Z}_5^{t_2} &= \Big\{((1),1),\ ((12345),\xi^2),\ ((13524),\xi^4),\ ((14253),\xi),\ ((15432),\xi^3)\Big\},\\
\mathbb{Z}_3^{t} &= \Big\{((1),1),\ ((123),\gamma),\ ((132),\gamma^2)\Big\},\\
D_3^z &= \Big\{((1),1),\ ((123),1),\ ((132),1),\ ((12)(45),-1),\ ((13)(45),-1),\\
&\qquad ((23)(45),-1)\Big\},\\
A_4^{t_1} &= \Big\{((1),1),((12)(34),1),((13)(24),1),((14)(23),1),((123),\gamma),((132),\\
&\qquad \gamma^2),((124),\gamma^2),((142),\gamma),((134),\gamma),((143),\gamma^2),((234),\gamma^2),((243),\gamma)\Big\},\\
A_4^{t_2} &= \Big\{((1),1),((12)(34),1),((13)(24),1),((14)(23),1),((123),\gamma^2),((132),\\
&\qquad \gamma),((124),\gamma),((142),\gamma^2),((134),\gamma^2),((143),\gamma),((234),\gamma),((243),\gamma^2)\Big\},\\
D_5^z &= \Big\{((1),1),\ ((12345),1),\ ((13524),1),\ (14253),1),\ ((15432),1),\\
&\qquad ((12)(35),-1),\ ((13)(45),-1),\ ((14)(23),-1),\ ((15)(24),-1),\\
&\qquad ((25)(34),-1)\Big\},
\end{aligned}$$

where $\xi = e^{\frac{2\pi}{5}i}$, $\gamma = e^{\frac{2\pi i}{3}}$. The lattice of the conjugacy classes of the twisted subgroups in $A_5 \times S^1$ is shown in Figure A2.9.

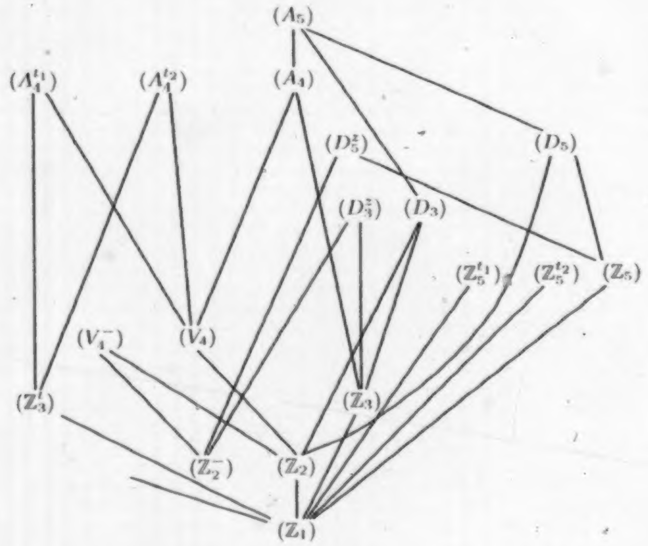
(A_5)
$(A_4^{t_1})$
$(A_4^{t_2})$
(A_4)
(D_5^z)
(D_5)
(D_3^z)
(D_3)
$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_5^{t_2})$
$(\mathbb{Z}_5)$
(V_4^-)
(V_4)
$(\mathbb{Z}_3^t)$
$(\mathbb{Z}_3)$
$(\mathbb{Z}_2^-)$
$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$

Fig. A2.9. Conjugacy classes of twisted subgroups in $A_5 \times S^1$

A2.1.6 Orthogonal Group $O(2)$

Denote by $O(2)$ be the *orthogonal group* of degree 2 over reals, which is defined as a subgroup in the general linear group $GL(2;\mathbb{R})$ by

$$O(2) = \{A \in GL(2;\mathbb{R}) \ : \ AA^T = I\},$$

where A^T is the transpose of A.

The subgroups in $O(2)$ include $O(2)$, $SO(2)$, D_n (for $n \in \mathbb{N}$), and $\mathbb{Z}_m$ (for $m \in \mathbb{N}$). Moreover, we have that

$$\Phi_0(O(2)) = \{(O(2)), (SO(2)), (D_n),\ n \in \mathbb{N}\},$$
$$\Phi_1(O(2)) = \{\mathbb{Z}_m,\ m \in \mathbb{N}\}.$$

The twisted one-folded subgroups in $O(2) \times S^1$ are

$$
\begin{aligned}
&O(2)^{-} := O(2)^{\varphi}, \quad \varphi : O(2) \to \mathbb{Z}_2, \quad \varphi(e^{i\theta}) = 1 \text{ and } \varphi(\kappa e^{i\theta}) = -1,\\
&SO(2)^{\varphi_k}, \quad \varphi_k : SO(2) \to S^1, \ \varphi_k(e^{i\theta}) = e^{ik\theta}, \ \ k \in \mathbb{N},\\
&D_k^z := D_k^{\psi}, \quad \psi : D_k \to \mathbb{Z}_2, \ \ker \psi = \mathbb{Z}_k,\\
&D_{2k}^d := D_{2k}^{\phi}, \quad \phi : D_{2k} \to \mathbb{Z}_2, \ \ker \phi = D_k.
\end{aligned}
$$

The lattice of the conjugacy classes of twisted subgroups in $O(2) \times S^1$ is shown on Figure A2.10.

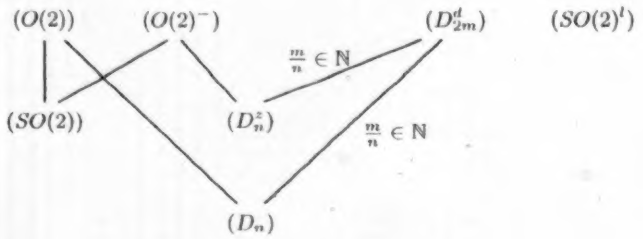

Fig. A2.10. Lattice of conjugacy classes of twisted subgroups in $O(2) \times S^1$

Furthermore,

$$
\begin{aligned}
\Phi_0(O(2) \times S^1) &= \{O(2) \times S^1, SO(2) \times S^1, D_n \times S^1, \ n \in \mathbb{N}\},\\
\Phi_1(O(2) \times S^1) &= \{\mathbb{Z}_m \times S^1, O(2) \times \mathbb{Z}_l, SO(2) \times \mathbb{Z}_l, D_n \times \mathbb{Z}_l\\
&\qquad O(2)^{-,l}, SO(2)^{\varphi_k,l}, D_k^{z,l}, D_{2k}^{d,l}, \ m, n, l \in \mathbb{N}\},\\
\Phi_2(O(2) \times S^1) &= \{\mathbb{Z}_m \times \mathbb{Z}_l, \mathbb{Z}_m^{\varphi_k,l}, \mathbb{Z}_{2k}^{d,l}, \ m, l \in \mathbb{N}\}.
\end{aligned}
$$

A2.1.7 Tori Group T^n

We write $T^N = T^{N-1} \times S^1$. There are two types of subgroups in T^N:

(i) those of the form $H \times K$, for $H \subset T^{N-1}$ and $K \subset S^1$;
(ii) the twisted subgroups $H^{\varphi,l}$, for $H \subset T^{N-1}$, $\varphi : H \to S^1$ and $l \in \mathbb{N}$.

Thus, the set of all subgroups in T^N can be obtained inductively from the set of all subgroups in T^{N-1}. For simplicity, we assume $N = 2$ and list all the subgroups in $T^2 \simeq SO(2) \times S^1$, namely

(a) $(\mathbb{Z}_n \times S^1)$, $SO(2) \times \mathbb{Z}_m$, $\mathbb{Z}_k \times \mathbb{Z}_l$, where $n, m, k, l \in \mathbb{N}$;
(b) $(SO(2)^{\varphi_n, l_1})$, $(\mathbb{Z}_m^{\varphi_n, l_2})$, where $\varphi_n : SO(2) \to S^1$, $z \mapsto z^n$, $l_1, l_2 \in \mathbb{N}$.

A2.2 Irreducible Representations of Groups

A2.2.1 Irreducible Representations of S^1

We list the irreducible representations of S^1 in Table A2.1.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	$\gamma x := x$, $\gamma \in S^1$, $x \in \mathbb{R}$	Trivial
${}^l\mathcal{V}$	$\mathbb{C}$	$\gamma z := \gamma^l \cdot z$, $\gamma \in S^1$, $z \in \mathbb{C}$	$l \in \mathbb{N}$

Table A2.1. Irreducible representations of S_1

A2.2.2 Irreducible Representations of T^n

Notice that all the nontrivial irreducible representations of an abelian group have a complex dimension 1. Thus, an irreducible T^n-representation $\mathcal{V}$ is a copy $\mathbb{C}$, with the T^n-action given by

$$(\gamma_1, \gamma_2, \cdots, \gamma_n)z = \gamma_1^{l_1} \cdot \gamma_2^{l_2} \cdots \gamma_n^{l_n} \cdot z,$$

where $\gamma_i \in S^1$, $l_i \in \mathbb{N}$ and " $\cdot$ " stands for the complex multiplication. Denote this irreducible representation by ${}^{(l_1,\cdots,l_n)}\mathcal{V}$.

A2.2.3 Irreducible Representations of Q_8 and $Q_8 \times S^1$

Let us list all the irreducible representations of Q_8 in Table A2.2 and all the 1-folded irreducible representations of $Q_8 \times S^1$ in Table A2.3.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	Trivial	
$\mathcal{V}_k$	$\mathbb{R}$	Induced by $\varphi_k : Q_8 \to \mathbb{Z}_2$, $\ker \varphi_k = \mathbb{Z}_4^k$	$k = 1, 2, 3$
$\mathcal{V}_4$	$\mathbb{R}^4$	Natural	

Table A2.2. Irreducible representations of Q_8

$\mathcal{V}_{j,1}$	Space	Group Actions	Remarks
$\mathcal{V}_{0,1}$	$\mathbb{C}$	Trivial	
$\mathcal{V}_{k,1}$	$\mathbb{C}$	Induced by $\varphi_k : Q_8 \to \mathbb{Z}_2$, $\ker \varphi_k = \mathbb{Z}_4^k$	$k = 1, 2, 3$
$\mathcal{V}_{4,1}$	$\mathbb{C}^4$	Natural	

Table A2.3. Irreducible representations of $Q_8 \times S^1$

A2.2.4 Irreducible Representations of D_N and $D_N \times S^1$

We list all the irreducible representations of D_N in Table A2.4 and all the 1-folded irreducible representations of $D_N \times S^1$ in Table A2.5.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	Trivial	
$\mathcal{V}_j$	$\mathbb{C}$	$\begin{cases} \gamma z := \gamma^j \cdot z, \\ \kappa z := \bar{z}, \end{cases} \quad \gamma \in \mathbb{Z}_N, z \in \mathbb{C}$	$1 \leq j < N/2$
$\mathcal{V}_{j_N}$	$\mathbb{R}$	Induced by $\varphi : D_N \to \mathbb{Z}_2$, $\ker \varphi = \mathbb{Z}_N$	$j_N := \lfloor (N+1)/2 \rfloor$
$\mathcal{V}_{j_N+1}$	$\mathbb{R}$	Induced by $\varphi : D_N \to \mathbb{Z}_2$, $\ker \varphi = D_{N/2}$	N even
$\mathcal{V}_{j_N+2}$	$\mathbb{R}$	Induced by $\varphi : D_N \to \mathbb{Z}_2$, $\ker \varphi = \widetilde{D}_{N/2}$	N even

Table A2.4. Irreducible representations of D_N

$\mathcal{V}_{j,1}$	Space	Group Actions	Remarks
$\mathcal{V}_{0,1}$	$\mathbb{C}$	Trivial	
$\mathcal{V}_{j,1}$	$\mathbb{C}^2$	$\begin{cases} \gamma(z_1, z_2) := (\gamma^j \cdot z_1, \gamma^{-j} \cdot z_2, \\ \kappa(z_1, z_2) := (z_2, z_1), \end{cases} \quad \gamma \in \mathbb{Z}_N, z_1, z_2 \in \mathbb{C}$	$1 \leq j < N/2$
$\mathcal{V}_{j_N,1}$	$\mathbb{C}$	Induced by $\varphi : D_N \to \mathbb{Z}_2$, $\ker \varphi = \mathbb{Z}_N$	$j_N := \lfloor (N+1)/2 \rfloor$
$\mathcal{V}_{j_N+1,1}$	$\mathbb{C}$	Induced by $\varphi : D_N \to \mathbb{Z}_2$, $\ker \varphi = D_{N/2}$	N even
$\mathcal{V}_{j_N+2,1}$	$\mathbb{C}$	Induced by $\varphi : D_N \to \mathbb{Z}_2$, $\ker \varphi = \widetilde{D}_{N/2}$	N even

Table A2.5. Irreducible representations of $D_N \times S^1$

A2.2.5 Irreducible Representations of A_4 and $A_4 \times S^1$

Let us list all the irreducible representations of A_4 in Table A2.6 and all the 1-folded irreducible representations of $A_4 \times S^1$ in Table A2.7.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	Trivial	
$\mathcal{V}_2$	$\mathbb{C}$	Induced by $\varphi : A_4 \to \mathbb{Z}_3$, $\ker \varphi = V_4$	
$\mathcal{V}_3$	$\mathbb{R}^3$	Natural	

Table A2.6. Irreducible representations of A_4

$\mathcal{V}_{j,1}$	Space	Group Actions	Remarks
$\mathcal{V}_{0,1}$	$\mathbb{C}$	Trivial	
$\mathcal{V}_{j,1}$	$\mathbb{C}^2$	Induced by $\varphi_j : A_4 \xrightarrow{\varphi} \mathbb{Z}_3 \xrightarrow{\gamma\mapsto\gamma^2} \mathbb{Z}_3$	$j = 1, 2$
$\mathcal{V}_{3,1}$	$\mathbb{C}^3$	Natural	

Table A2.7. Irreducible representations of $A_4 \times S^1$

A2.2.6 Irreducible Representations of S_4 and $S_4 \times S^1$

We list all the irreducible representations of S_4 in Table A2.8 and all the 1-folded irreducible representations of $S_4 \times S^1$ in Table A2.9.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	Trivial	
$\mathcal{V}_1$	$\mathbb{R}$	Induced by $\varphi : S_4 \to \mathbb{Z}_2$, $\ker \varphi = A_4$	
$\mathcal{V}_2$	$\mathbb{C}$	Induced by $\varphi : S_4 \to S_3 \simeq D_3$, $\ker \varphi = V_4$	
$\mathcal{V}_3$	$\mathbb{R}^3$	Natural	
$\mathcal{V}_4$	$\mathcal{V}_1 \otimes \mathcal{V}_3$	Natural 3-dim rep. with nontrivial 1-dim rep.	

Table A2.8. Irreducible representations of S_4

$\mathcal{V}_{j,1}$	Space	Group Actions	Remarks
$\mathcal{V}_{0,1}$	$\mathbb{C}$	Trivial	
$\mathcal{V}_{1,1}$	$\mathbb{C}$	Induced by $\varphi : S_4 \to \mathbb{Z}_2$, $\ker \varphi = A_4$	
$\mathcal{V}_{2,1}$	$\mathbb{C}^2$	Induced by $\varphi : S_4 \to S_3 \simeq D_3$, $\ker \varphi = V_4$	
$\mathcal{V}_{3,1}$	$\mathbb{C}^3$	Natural	
$\mathcal{V}_{4,1}$	$\mathcal{V}_{1,1} \otimes \mathcal{V}_{3,1}$	Natural 3-dim rep. with nontrivial 1-dim rep.	

Table A2.9. Irreducible representations of $S_4 \times S^1$

A2.2.7 Irreducible Representations of A_5 and $A_5 \times S^1$

Let us list all the irreducible representations of A_5 in Table A2.10 and all the 1-folded irreducible representations of $A_5 \times S^1$ in Table A2.11.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	Trivial	
$\mathcal{V}_1$	$\mathbb{R}^4$	Natural	
$\mathcal{V}_2$	$\mathbb{R}^5$	Spherical harmonics of 3 variables	$A_5 \subset SO(3)$
$\mathcal{V}_3$	$\mathbb{R}^3$	Character $\chi((12345)) = \frac{1+\sqrt{5}}{2}$	
$\mathcal{V}_4$	$\mathbb{R}^3$	Character $\chi((12345)) = \frac{1-\sqrt{5}}{2}$	

Table A2.10. Irreducible representations of A_5

$\mathcal{V}_{j,1}$	Space	Group Actions	Remarks
$\mathcal{V}_{0,1}$	$\mathbb{C}$	Trivial	
$\mathcal{V}_{1,1}$	$\mathbb{C}^4$	Complexification $\mathcal{V}_1^c$ of $\mathcal{V}_1$	
$\mathcal{V}_{2,1}$	$\mathbb{C}^5$	Complexification $\mathcal{V}_2^c$ of $\mathcal{V}_2$	
$\mathcal{V}_{3,1}$	$\mathbb{C}^3$	Complexification $\mathcal{V}_3^c$ of $\mathcal{V}_3$	
$\mathcal{V}_{4,1}$	$\mathbb{C}^3$	Complexification $\mathcal{V}_4^c$ of $\mathcal{V}_4$	

Table A2.11. Irreducible representations of $A_5 \times S^1$

A2.2.8 Irreducible Representations of $O(2)$ and $O(2) \times S^1$

Let us list all the irreducible representations of $O(2)$ in Table A2.12 and all the 1-folded irreducible representations of $O(2) \times S^1$ in Table A2.13.

$\mathcal{V}_i$	Space	Group Actions	Remarks
$\mathcal{V}_0$	$\mathbb{R}$	Trivial	
$\mathcal{V}_{\frac{1}{2}}$	$\mathbb{R}$	Induced by $\varphi : O(2) \to \mathbb{Z}_2$, $\ker \varphi = SO(2)$	
$\mathcal{V}_m$	$\mathbb{C}$	$\begin{cases} uz := u^m \cdot z, \\ \kappa z := \bar{z}, \end{cases} \quad u \in O(2), z \in \mathbb{C}$	$m = 1, 2, 3, \dots$

Table A2.12. Irreducible representations of $O(2)$

$\mathcal{V}_{j,1}$	Space	Group Actions	Remarks
$\mathcal{V}_{0,1}$	$\mathbb{C}$	Trivial	
$\mathcal{V}_{\frac{1}{2},1}$	$\mathbb{C}$	Induced by $\varphi : O(2) \to \mathbb{Z}_2$, $\ker \varphi = SO(2)$	
$\mathcal{V}_{m,1}$	$\mathbb{C}^2$	$\begin{cases} uz := u^m \cdot z, \\ \kappa z := \bar{z}, \end{cases} \quad u \in O(2), z \in \mathbb{C}$	$m = 1, 2, 3, \dots$

Table A2.13. Irreducible representations of $O(2) \times S^1$

A2.3 Basic Degrees for Groups

The concept of basic degrees plays an important role in the effective computations of $\Gamma \times S^1$-equivariant degrees. In this section, we catalog the values of all the basic degrees in the case $\Gamma = Q_8$, D_N, A_4, S_4, A_5, and $O(2)$. For more details, we refer to [15].

A2.3.1 Basic Degrees for Q_8

For convenience, we present the lattice of twisted orbit types in $\mathcal{V}_{4,1}$ in Figure A2.11. Based on the lattices of orbit types occured in the irreducible representations, we obtain the basic degrees of the irreducible representations of Q_8 and $Q_8 \times S^1$ respectively.

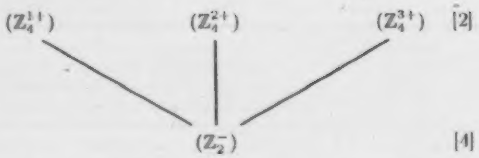
$(\mathbb{Z}_4^{1+})$
$(\mathbb{Z}_4^{2+})$
$(\mathbb{Z}_4^{3+})$
[2]
$(\mathbb{Z}_2^-)$
[4]

Fig. A2.11. Lattice of twisted orbit types in $\mathcal{V}_{4,1}$

$$\begin{cases} \deg_{\mathcal{V}_0} = -(Q_8), \\ \deg_{\mathcal{V}_k} = (Q_8) - (\mathbb{Z}_4^k), \\ \deg_{\mathcal{V}_4} = (Q_8). \end{cases} \qquad \begin{cases} \deg_{\mathcal{V}_{0,1}} = (Q_8), \\ \deg_{\mathcal{V}_{k,1}} = (Q_8^{k-}), \\ \deg_{\mathcal{V}_{4,1}} = (\mathbb{Z}_4^{1+}) + (\mathbb{Z}_4^{2+}) + (\mathbb{Z}_4^{3+}) - (\mathbb{Z}_2^-); \end{cases}$$

where $k = 1, 2, 3$.

A2.3.2 Basic Degrees for D_N

The lattices of twisted orbit types for $\mathcal{V}_{j,1}$ are listed in Figure A2.12— Figure A2.14. Based on the lattices of orbit types, we obtain the basic degrees of irreducible representations for D_N and $D_N \times S^1$ respectively.

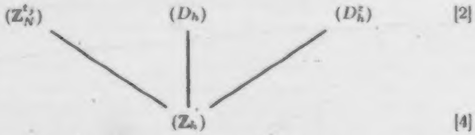
$(\mathbb{Z}_N^{t_j})$
(D_h)
(D_h^z)
[2]
$(\mathbb{Z}_h)$
[4]

Fig. A2.12. Lattice of twisted orbit types for m Odd

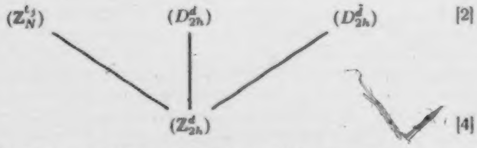
$(\mathbb{Z}_N^{t_j})$
(D_{2h}^d)
$(D_{2h}^{\hat{d}})$
[2]
$(\mathbb{Z}_{2h}^d)$
[4]

Fig. A2.13. Lattice of twisted orbit types for $m \equiv 2 \pmod 4$

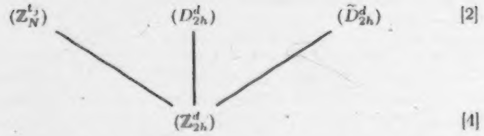

Fig. A2.14. Lattice of twisted orbit types for $m \equiv 0 \pmod 4$

$$\begin{cases} \deg_{\mathcal{V}_0} = -(D_N), \\ \deg_{\mathcal{V}_j} = \begin{cases} (D_N) - 2(D_h) + (\mathbb{Z}_h) & \text{if } m \text{ is odd,} \\ (D_N) - (D_h) - (\widetilde{D}_h) + (\mathbb{Z}_h) & \text{if } m \text{ is even,} \end{cases} \\ \deg_{\mathcal{V}_{j_N}} = (D_N) - (\mathbb{Z}_N), \\ \deg_{\mathcal{V}_{j_N+1}} = (D_N) - (D_{\frac{N}{2}}), \quad \text{if } N \text{ is even,} \\ \deg_{\mathcal{V}_{j_N+2}} = (D_N) - (\widetilde{D}_{\frac{N}{2}}), \quad \text{if } N \text{ is even,} \end{cases}$$

where $1 \leq j < N/2$, $h = \gcd(j, N)$ and $m := N/h$.

$$\begin{cases} \deg_{\mathcal{V}_{0,1}} = (D_N), \\ \deg_{\mathcal{V}_{j,1}} = \begin{cases} (\mathbb{Z}_N^{t_j}) + (D_h) + (D_h^z) - (\mathbb{Z}_h) & \text{if } m \text{ is odd,} \\ (\mathbb{Z}_N^{t_j}) + (D_{2h}^d) + (D_{2h}^{\hat{d}}) - (\mathbb{Z}_{2h}^d) & \text{if } m \equiv 2 \pmod 4, \\ (\mathbb{Z}_N^{t_j}) + (D_{2h}^d) + (\widetilde{D}_{2h}^d) - (\mathbb{Z}_{2h}^d) & \text{if } m \equiv 0 \pmod 4 \end{cases} \\ \deg_{\mathcal{V}_{j_N,1}} = (D_N^z), \\ \deg_{\mathcal{V}_{j_N+1,1}} = (D_N^d), \quad \text{if } N \text{ is even,} \\ \deg_{\mathcal{V}_{j_N+2,1}} = (D_N^{\hat{d}}), \quad \text{if } N \text{ is even,} \end{cases}$$

where $1 \leq j < N/2$, $h := \gcd(j, N)$ and $m := N/h$.

A2.3.3 Basic Degrees for A_4

We list the lattices of the twisted orbit types in $\mathcal{V}_3$ and $\mathcal{V}_{3,1}$ in Figure A2.15. Based on the lattices of orbit types, we obtain the basic degrees of irreducible representations for A_4 and $A_4 \times S^1$ respectively.

$$\begin{cases} \deg_{\mathcal{V}_0} = -(A_4), \\ \deg_{\mathcal{V}_2} = (A_4), \\ \deg_{\mathcal{V}_3} = (A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1). \end{cases}$$

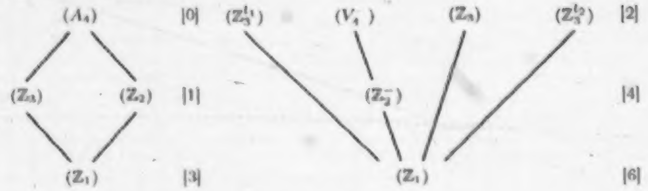
(A_4)
[0]
$(\mathbb{Z}_3)$
$(\mathbb{Z}_2)$
[1]
$(\mathbb{Z}_1)$
[3]
$(\mathbb{Z}_3^{t_1})$
(V_4^-)
$(\mathbb{Z}_3)$
$(\mathbb{Z}_3^{t_2})$
[2]
$(\mathbb{Z}_2^-)$
[4]
$(\mathbb{Z}_1)$
[6]

Fig. A2.15. Representation $\mathcal{V}_3$ and representation $\mathcal{V}_{3,1}$

$$\begin{cases} \deg_{\mathcal{V}_{0,1}} = (A_4), \\ \deg_{\mathcal{V}_{1,1}} = (A_4^{t_1}), \\ \deg_{\mathcal{V}_{2,1}} = (A_4^{t_2}), \\ \deg_{\mathcal{V}_{3,1}} = (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (V_4^-) + (\mathbb{Z}_3) - (\mathbb{Z}_1). \end{cases}$$

A2.3.4 Basic Degrees for S_4

We list the lattices of the twisted orbit types in $\mathcal{V}_{2,1}$, $\mathcal{V}_{3,1}$ and $\mathcal{V}_{4,1}$ in Figure A2.16 — Figure A2.18. Based on the lattices of orbit types, we obtain the basic degrees of irreducible representations for S_4 and $S_4 \times S^1$ respectively.

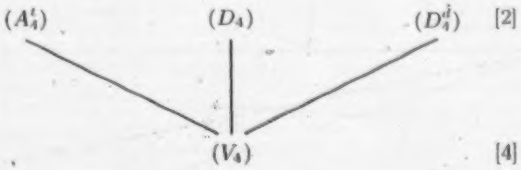
(A_4^t)
(D_4)
(D_4^d)
[2]
(V_4)
[4]

Fig. A2.16. Lattice of twisted orbit types for $\mathcal{V}_{2,1}$

$$\begin{cases} \deg_{\mathcal{V}_0} = -(S_4), \\ \deg_{\mathcal{V}_1} = (S_4) - (A_4), \\ \deg_{\mathcal{V}_2} = (S_4) - 2(D_4) + (V_4), \\ \deg_{\mathcal{V}_3} = (S_4) - 2(D_3) - (D_2) + 3(D_1) - (\mathbb{Z}_1), \\ \deg_{\mathcal{V}_4} = (S_4) - (\mathbb{Z}_4) - (D_1) - (\mathbb{Z}_3) + (\mathbb{Z}_1). \end{cases}$$

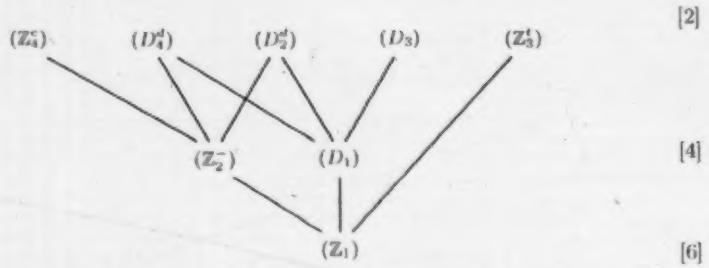
[2]
$(\mathbb{Z}_4^c)$
(D_4^d)
(D_2^d)
(D_3)
$(\mathbb{Z}_3^t)$
[4]
$(\mathbb{Z}_2^-)$
(D_1)
[6]
$(\mathbb{Z}_1)$

Fig. A2.17. Lattice of twisted orbit types for $\mathcal{V}_{3,1}$

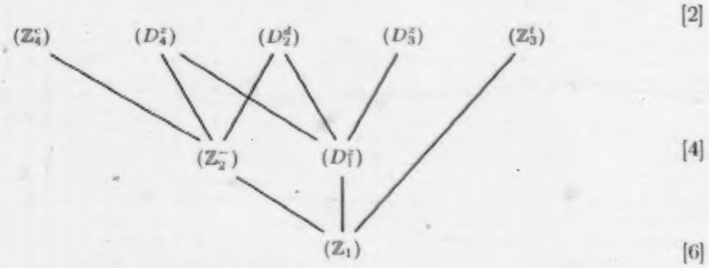
[2]
$(\mathbb{Z}_4^c)$
(D_4^z)
(D_2^d)
(D_3^z)
$(\mathbb{Z}_3^t)$
[4]
$(\mathbb{Z}_2^-)$
(D_1^z)
[6]
$(\mathbb{Z}_1)$

Fig. A2.18. Lattice of twisted orbit types for $\mathcal{V}_{4,1}$

$$
\begin{cases}
\deg_{\mathcal{V}_{0,1}} = (S_4), \\
\deg_{\mathcal{V}_{1,1}} = (S_4^-), \\
\deg_{\mathcal{V}_{2,1}} = (A_4^t) + (D_4) + (D_4^d) - (V_4), \\
\deg_{\mathcal{V}_{3,1}} = (\mathbb{Z}_4^c) + (D_4^d) + (D_2^d) + (D_3) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1), \\
\deg_{\mathcal{V}_{4,1}} = (\mathbb{Z}_4^c) + (D_4^z) + (D_2^d) + (D_3^z) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1^z).
\end{cases}
$$

A2.3.5 Basic Degrees for A_5

We list the lattices of the twisted orbit types in $\mathcal{V}_k$ and $\mathcal{V}_{k,1}$ for $k = 1, 2, 3, 4$ respectively in Figure A2.19 — Figure A2.23. Based on the lattices of orbit types, we obtain the basic degrees of irreducible representations for A_5 and $A_5 \times S^1$ respectively.

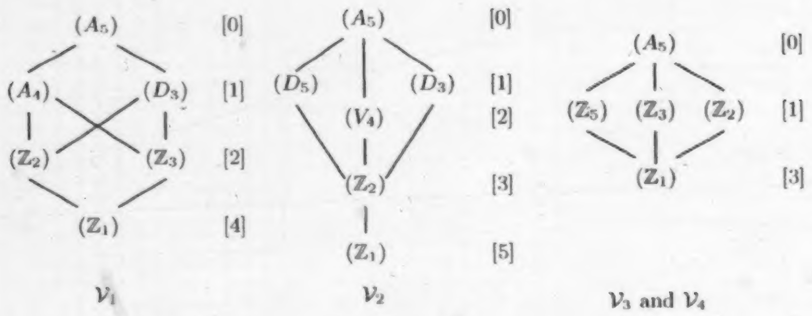
(A_5)
[0]
(A_4)
(D_3)
[1]
$(\mathbb{Z}_2)$
$(\mathbb{Z}_3)$
[2]
$(\mathbb{Z}_1)$
[4]
$\mathcal{V}_1$
(A_5)
[0]
(D_5)
(D_3)
[1]
(V_4)
[2]
$(\mathbb{Z}_2)$
[3]
$(\mathbb{Z}_1)$
[5]
$\mathcal{V}_2$
(A_5)
[0]
$(\mathbb{Z}_5)$
$(\mathbb{Z}_3)$
$(\mathbb{Z}_2)$
[1]
$(\mathbb{Z}_1)$
[3]
$\mathcal{V}_3$ and $\mathcal{V}_4$

Fig. A2.19. Lattice of orbit types for $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$

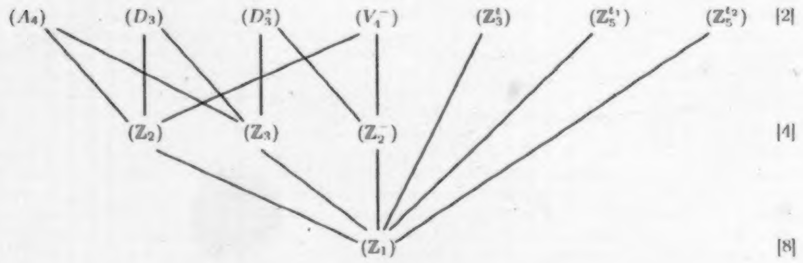
(A_4)
(D_3)
(D_3^z)
(V_4^-)
$(\mathbb{Z}_3^t)$
$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_5^{t_2})$
[2]
$(\mathbb{Z}_2)$
$(\mathbb{Z}_3)$
$(\mathbb{Z}_2^-)$
[4]
$(\mathbb{Z}_1)$
[8]

Fig. A2.20. Lattice of twisted orbit types for $\mathcal{V}_{1,1}$

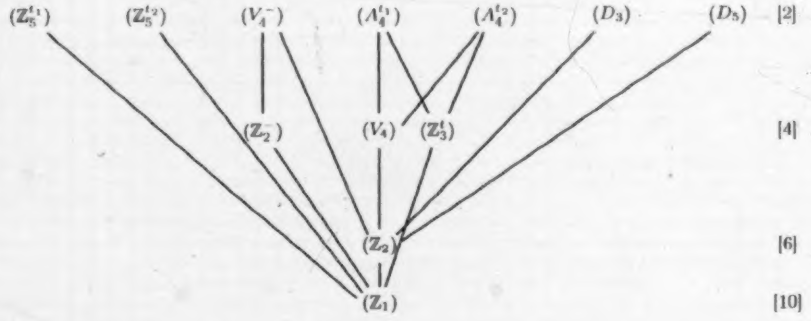
$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_5^{t_2})$
(V_4^-)
$(A_4^{t_1})$
$(A_4^{t_2})$
(D_3)
(D_5)
[2]
$(\mathbb{Z}_2^-)$
(V_4)
$(\mathbb{Z}_3^t)$
[4]
$(\mathbb{Z}_2)$
[6]
$(\mathbb{Z}_1)$
[10]

Fig. A2.21. Lattice of twisted orbit types for $\mathcal{V}_{2,1}$

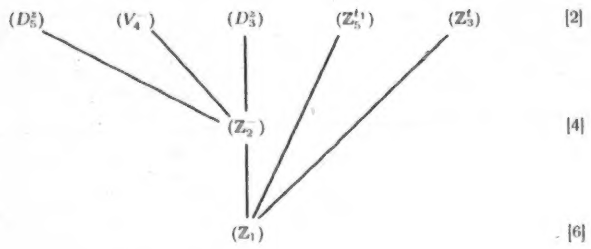
(D_5^z)
(V_4^-)
(D_3^z)
$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_3^t)$
[2]
$(\mathbb{Z}_2^-)$
[4]
$(\mathbb{Z}_1)$
[6]

Fig. A2.22. Lattice of twisted orbit types for $\mathcal{V}_{3,1}$

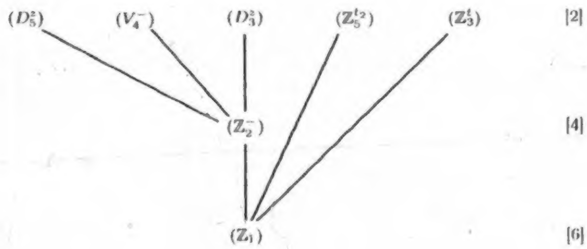
(D_5^z)
(V_4^-)
(D_3^z)
$(\mathbb{Z}_5^{t_2})$
$(\mathbb{Z}_3^t)$
[2]
$(\mathbb{Z}_2^-)$
[4]
$(\mathbb{Z}_1)$
[6]

Fig. A2.23. Lattice of twisted orbit types for $\mathcal{V}_{4,1}$

$$\begin{cases} \deg_{\mathcal{V}_0} = -(A_5), \\ \deg_{\mathcal{V}_1} = (A_5) - 2(A_4) - 2(D_3) + 3(\mathbb{Z}_2) + 3(\mathbb{Z}_3) - 2(\mathbb{Z}_1), \\ \deg_{\mathcal{V}_2} = (A_5) - 2(D_5) - 2(D_3) + 3(\mathbb{Z}_2) - (\mathbb{Z}_1), \\ \deg_{\mathcal{V}_3} = \deg_{\mathcal{V}_4} = (A_5) - (\mathbb{Z}_5) - (\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1). \end{cases}$$

$$\begin{cases} \deg_{\mathcal{V}_{0,1}} = (A_5), \\ \deg_{\mathcal{V}_{1,1}} = (A_4) + (D_3) + (D_3^z) + (V_4^-) + (\mathbb{Z}_3^t) \\ \qquad +(\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_2) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-), \\ \deg_{\mathcal{V}_{2,1}} = (D_5) + (D_3) + (A_4^{t_1}) + (A_4^{t_2}) \\ \qquad +(V_4^-) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - 2(\mathbb{Z}_2), \\ \deg_{\mathcal{V}_{3,1}} = (D_5^z) + (V_4^-) + (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-), \\ \deg_{\mathcal{V}_{4,1}} = (D_5^z) + (V_4^-) + (D_3^z) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-). \end{cases}$$

A2.3.6 Basic Degrees for $O(2)$

We list the basic degrees of irreducible representations for $O(2)$ and $O(2)\times S^1$ respectively.

$$\begin{cases} \deg_{\mathcal{V}_0} = (O(2)), \\ \deg_{\mathcal{V}_{\frac{1}{2}}} = (O(2)) - (SO(2)), \\ \deg_{\mathcal{V}_m} = (O(2)) - (D_m), \quad m = 1,2,3,\dots \end{cases}$$

$$\begin{cases} \deg_{\mathcal{V}_{0,1}} = (O(2)), \\ \deg_{\mathcal{V}_{\frac{1}{2},1}} = (O(2)^-), \\ \deg_{\mathcal{V}_{m,1}} = (SO(2)^m) + (D^d_{2m}), \quad m = 1,2,3,\dots. \end{cases}$$

A2.3.7 Basic Gradient Degrees for $O(2)\times S^1$

$$\begin{cases} \mathrm{Deg}_{\mathcal{V}_0} & = (O(2)\times S^1), \\ \mathrm{Deg}_{\mathcal{V}_{\frac{1}{2}}} & = (O(2)\times S^1) - (SO(2)\times S^1), \\ \mathrm{Deg}_{\mathcal{V}_m} & = (O(2)\times S^1) - (D_m\times S^1), \end{cases}$$

$$\begin{cases} \mathrm{Deg}_{\mathcal{V}_{0,1}} & = (O(2)\times S^1) - (O(2)), \\ \mathrm{Deg}_{\mathcal{V}_{\frac{1}{2},1}} & = (O(2)\times S^1) - (O(2)^-), \\ \mathrm{Deg}_{\mathcal{V}_{m,1}} & = (O(2)\times S^1) - (SO(2)^{\varphi_m}) - (D^d_{2m}) + (\mathbb{Z}^d_{2m}), \end{cases}$$

where $m = 1,2,\dots$.

A3

Multiplication Tables

For convenience, we present the multiplication tables for the Burnside ring $A(\Gamma)$, the $A(\Gamma)$-module $A_1^t(\Gamma \times S^1)$, for $\Gamma = Q_8$, D_n (for $n = 3, 4, 5, 6$), A_4, S_4, A_5, and $O(2)$. In addition, we include the multiplication tables for the Euler ring $U(T^2)$ and $U(O(2) \times S^1)$.

A3.1 Multiplication Tables for the Burnside Ring $A(\Gamma)$

	(Q_8)	$(\mathbb{Z}_4^1)$	$(\mathbb{Z}_4^2)$	$(\mathbb{Z}_4^3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(Q_8)	$(\mathbb{Z}_4^1)$	$(\mathbb{Z}_4^2)$	$(\mathbb{Z}_4^3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(Q_8)
$(\mathbb{Z}_4^1)$	$2(\mathbb{Z}_4^1)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^1)$
$(\mathbb{Z}_4^2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^2)$
$(\mathbb{Z}_4^3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^3)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^3)$
$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.1. Multiplication table for the Burnside ring $A(Q_8)$

	(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$
(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	(D_3)
(D_1)	$(D_1) + (\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_1)
$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.2. Multiplication table for the Burnside ring $A(D_3)$

	(D_4)	(D_2)	$(\widetilde{D}_2)$	(D_1)	$(\widetilde{D}_1)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_4)	(D_2)	$(\widetilde{D}_2)$	(D_1)	$(\widetilde{D}_1)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_4)
(D_2)	$2(D_2)$	$(\mathbb{Z}_2)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	(D_2)
$(\widetilde{D}_2)$	$(\mathbb{Z}_2)$	$2(\widetilde{D}_2)$	$(\mathbb{Z}_1)$	$2(\widetilde{D}_1)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\widetilde{D}_2)$
(D_1)	$2(D_1)$	$(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_1)
$(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$2(\widetilde{D}_1)$	$2(\mathbb{Z}_1)$	$2(\widetilde{D}_1)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\widetilde{D}_1)$
$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_4)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4)$
$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.3. Multiplication table for the Burnside ring $A(D_4)$

	(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$
(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	(D_5)
(D_1)	$(D_1)+2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	(D_1)
$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_5)$
$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.4. Multiplication table for the Burnside ring $A(D_5)$

	(D_6)	$(\widetilde{D}_3)$	(D_3)	$(\mathbb{Z}_6)$	(D_2)	$(\mathbb{Z}_3)$	$(\widetilde{D}_1)$	(D_1)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_6)	$(\widetilde{D}_3)$	(D_3)	$(\mathbb{Z}_6)$	(D_2)	$(\mathbb{Z}_3)$	$(\widetilde{D}_1)$	(D_1)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_6)
$(\widetilde{D}_3)$	$2(\widetilde{D}_3)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\widetilde{D}_1)$	$3(\mathbb{Z}_3)$	$2(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\widetilde{D}_3)$
(D_3)	$(\mathbb{Z}_3)$	$2(D_3)$	$(\mathbb{Z}_3)$	(D_1)	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	(D_3)
$(\mathbb{Z}_6)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_6)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_6)$
(D_2)	$(\widetilde{D}_1)$	(D_1)	$(\mathbb{Z}_2)$	$(D_2)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1)+(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	(D_2)
$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$4(\mathbb{Z}_3)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\widetilde{D}_1)$	$2(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\widetilde{D}_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\widetilde{D}_1)$
(D_1)	$(\mathbb{Z}_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_1)
$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.5. Multiplication table for the Burnside ring $A(D_6)$

	(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(A_4)
(V_4)	$3(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	(V_4)
$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.6. Multiplication table for the Burnside ring $A(A_4)$

	(A_4)	(D_4)	(D_3)	(D_2)	(V_4)	$(\mathbb{Z}_4)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	(D_1)	$(\mathbb{Z}_1)$
$2(A_4)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	(A_4)
(V_4)	$(D_4)+(V_4)$	(D_1)	$(D_2)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_4)$	$3(\mathbb{Z}_2)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_4)
$(\mathbb{Z}_3)$	(D_1)	$(D_3)+(D_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_3)
$(\mathbb{Z}_2)$	$(D_2)+(\mathbb{Z}_2)$	$2(D_1)$	$2(D_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_2)
$2(V_4)$	$3(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$6(V_4)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(V_4)
$(\mathbb{Z}_2)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_4)$
$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$2(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$2(D_1)+5(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	(D_1)
$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$24(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.7. Multiplication table for the Burnside ring $A(S_4)$

	(A_5)	(A_4)	(D_5)	(D_3)	$(\mathbb{Z}_5)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(A_5)	(A_4)	(D_5)	(D_3)	$(\mathbb{Z}_5)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(A_5)
(A_4)	$(A_4)+(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_3)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(V_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	(A_4)
(D_5)	$(\mathbb{Z}_2)$	$(D_5)+(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_5)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_5)
(D_3)	$(\mathbb{Z}_3)+(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(D_3)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+3(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	(D_3)
$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_5)$
(V_4)	$(V_4)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(V_4)+3(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)+6(\mathbb{Z}_1)$	$15(\mathbb{Z}_1)$	(V_4)
$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$20(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)+6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+14(\mathbb{Z}_1)$	$30(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$15(\mathbb{Z}_1)$	$20(\mathbb{Z}_1)$	$30(\mathbb{Z}_1)$	$60(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$

Table A3.8. Multiplication table for the Burnside ring $A(A_5)$

	$(O(2))$	$(SO(2))$	(D_m)
$(O(2))$	$(SO(2))$	(D_m)	$(O(2))$
$(SO(2))$	$2(SO(2))$	0	$(SO(2))$
(D_n)	0	$2(D_l)$, where $l = \gcd(n, m)$	(D_n)

Table A3.9. Multiplication table for the Burnside ring $A(O(2))$

A3.2 Multiplication Tables for the $A(\Gamma)$-Module $A_1^t(\Gamma \times S^1)$

	(Q_8)	$(\mathbb{Z}_4^1)$	$(\mathbb{Z}_4^2)$	$(\mathbb{Z}_4^3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(Q_8^{1-})	$(\mathbb{Z}_4^1)$	$(\mathbb{Z}_4^{2-})$	$(\mathbb{Z}_4^{3-})$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(Q_8^{1-})
(Q_8^{2-})	$(\mathbb{Z}_4^{1-})$	$(\mathbb{Z}_4^2)$	$(\mathbb{Z}_4^{3-})$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(Q_8^{2-})
(Q_8^{3-})	$(\mathbb{Z}_4^{1-})$	$(\mathbb{Z}_4^{2-})$	$(\mathbb{Z}_4^3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(Q_8^{3-})
$(\mathbb{Z}_4^{1+})$	$2(\mathbb{Z}_4^{1+})$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^{1+})$
$(\mathbb{Z}_4^{2+})$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_4^{2+})$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^{2+})$
$(\mathbb{Z}_4^{3+})$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_4^{3+})$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^{3+})$
$(\mathbb{Z}_4^{1-})$	$2(\mathbb{Z}_4^{1-})$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^{1-})$
$(\mathbb{Z}_4^{2-})$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^{2-})$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^{2-})$
$(\mathbb{Z}_4^{3-})$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^{3-})$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^{3-})$
$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$4(\mathbb{Z}_2^-)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table A3.10. Multiplication table for the $A(Q_8)$-module $A_1^t(Q_8 \times S^1)$

	(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$
$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$
(D_3^z)	(D_1^z)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	(D_3^z)
(D_1^z)	$(D_1^z) + (\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_1^z)

Table A3.11. Multiplication table for the $A(D_3)$-module $A_1^t(D_3 \times S^1)$

	(D_4)	(D_2)	$(\tilde{D}_2)$	(D_1)	$(\tilde{D}_1)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_4^z)	(D_2^z)	$(\tilde{D}_2^z)$	(D_1^z)	$(\tilde{D}_1^z)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_4^z)
$(\tilde{D}_4^d)$	(D_2^z)	$(\tilde{D}_2)$	(D_1^z)	$(\tilde{D}_1)$	$(\mathbb{Z}_4^d)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\tilde{D}_4^d)$
(D_4^d)	(D_2)	$(\tilde{D}_2^z)$	(D_1)	$(\tilde{D}_1^z)$	$(\mathbb{Z}_4^d)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_4^d)
(D_2^d)	$2(D_2^d)$	$(\mathbb{Z}_2^-)$	$(D_1) + (D_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	(D_2^d)
$(\tilde{D}_2^d)$	$(\mathbb{Z}_2^-)$	$2(\tilde{D}_2^d)$	$(\mathbb{Z}_1)$	$(\tilde{D}_1) + (\tilde{D}_1^z)$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\tilde{D}_2^d)$
(D_2^z)	$2(D_2^z)$	$(\mathbb{Z}_2)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	(D_2^z)
$(\tilde{D}_2^z)$	$(\mathbb{Z}_2)$	$2(\tilde{D}_2^z)$	$(\mathbb{Z}_1)$	$2(\tilde{D}_1^z)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\tilde{D}_2^z)$
(D_1^z)	$2(D_1^z)$	$(\mathbb{Z}_1)$	$2(D_1^z) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_1^z)
$(\tilde{D}_1^z)$	$(\mathbb{Z}_1)$	$2(\tilde{D}_1^z)$	$2(\mathbb{Z}_1)$	$2(\tilde{D}_1^z) + (\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\tilde{D}_1^z)$
$(\mathbb{Z}_4^t)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_4^t)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^t)$
$(\mathbb{Z}_4^d)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_4^d)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^d)$
$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$4(\mathbb{Z}_2^-)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table A3.12. Multiplication table for the $A(D_4)$-module $A_1^t(D_4 \times S^1)$

	(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$
$(\mathbb{Z}_5^{t_1})$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_1})$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_5^{t_2})$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_2})$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_2})$
(D_5^z)	(D_1^z)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	(D_5^z)
(D_1^z)	$(D_1^z) + 2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	(D_1^z)

Table A3.13. Multiplication table for the $A(D_5)$-module $A_1^t(D_5 \times S^1)$

	(D_6)	$(\tilde{D}_3)$	(D_3)	$(\mathbb{Z}_6)$	(D_2)	$(\mathbb{Z}_3)$	$(\tilde{D}_1)$	(D_1)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_6^z)	$(\tilde{D}_3^z)$	(D_3^z)	$(\mathbb{Z}_6)$	(D_2^z)	$(\mathbb{Z}_3)$	$(\tilde{D}_1^z)$	(D_1^z)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_6^z)
(D_6^d)	$(\tilde{D}_3)$	(D_3^z)	$(\mathbb{Z}_6^d)$	(D_2^d)	$(\mathbb{Z}_3)$	$(\tilde{D}_1)$	(D_1^z)	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	(D_6^d)
(D_6^d)	$(\tilde{D}_3^z)$	(D_3)	$(\mathbb{Z}_6^d)$	(D_2^d)	$(\mathbb{Z}_3)$	$(\tilde{D}_1^z)$	(D_1)	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	(D_6^d)
$(\mathbb{Z}_6^d)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_6^d)$	$(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_6^d)$
$(\mathbb{Z}_6^{t_1})$	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_6^{t_1})$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_6^{t_1})$
$(\mathbb{Z}_6^{t_2})$	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_6^{t_2})$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_6^{t_2})$
$(\tilde{D}_3^z)$	$2(\tilde{D}_3^z)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\tilde{D}_1^z)$	$2(\mathbb{Z}_3)$	$2(\tilde{D}_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\tilde{D}_3^z)$
(D_3^z)	$(\mathbb{Z}_3)$	$2(D_3^z)$	$(\mathbb{Z}_3)$	(D_1^z)	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	(D_3^z)
(D_2^z)	$(\tilde{D}_1^z)$	(D_1^z)	$(\mathbb{Z}_2)$	$(D_2^z)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\tilde{D}_1^z)+(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	(D_2^z)
(D_2^d)	$(\tilde{D}_1^z)$	(D_1)	$(\mathbb{Z}_2^-)$	$(D_2^d)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\tilde{D}_1^z)+(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$	(D_2^d)
(D_2^d)	$(\tilde{D}_1)$	(D_1^z)	$(\mathbb{Z}_2^-)$	$(D_2^d)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\tilde{D}_1)+(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$	(D_2^d)
$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$4(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$
$(\tilde{D}_1^z)$	$2(\tilde{D}_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\tilde{D}_1^z)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\tilde{D}_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\tilde{D}_1^z)$
(D_1^z)	$(\mathbb{Z}_1)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_1^z)
$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_2^-)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table A3.14. Multiplication table for the $A(D_6)$-module $A_1^t(D_6 \times S^1)$

	(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
$(A_4^{t_k})$	(V_4)	$(\mathbb{Z}_3^{t_k})$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(A_4^{t_k})$
(V_4^-)	$3(V_4^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$	(V_4^-)
$(\mathbb{Z}_3^{t_k})$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3^{t_k})+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_3^{t_k})$
$(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table A3.15. Multiplication table for the $A(A_4)$-module $A_1^t(A_4 \times S^1)$

	(A_4)	(D_4)	(D_3)	(D_2)	(V_4)	$(\mathbb{Z}_4)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	(D_1)	$(\mathbb{Z}_1)$
(A_4)	(D_4^d)	(D_3^z)	(D_2^z)	(V_4)	$(\mathbb{Z}_4^-)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	(D_1^z)	$(\mathbb{Z}_1)$	(S_4^-)
$2(A_4^t)$	(V_4)	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	(A_4^t)
(V_4^-)	$(D_4^z)+(V_4^-)$	(D_1^z)	$(D_2^z)+(\mathbb{Z}_2^-)$	$3(V_4^-)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_4^z)
(V_4^-)	$(D_4^d)+(V_4^-)$	(D_1)	$(D_2)+(\mathbb{Z}_2^-)$	$3(V_4^-)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_4^d)
(V_4)	$(D_4^{\hat{d}})+(V_4)$	(D_1^z)	$(D_2^z)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$(D_4^{\hat{d}})$
$(\mathbb{Z}_3)$	(D_1^z)	$(D_3^z)+(D_1^z)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_3^z)
$(\mathbb{Z}_2)$	$(D_2^z)+(\mathbb{Z}_2)$	$2(D_1^z)$	$2(D_2^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_2^z)
$(\mathbb{Z}_2^-)$	$(D_2^d)+(\mathbb{Z}_2^-)$	$(D_1)+(D_1^z)$	$2(D_2^d)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$(D_1)+(D_1^z)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_2^d)
$2(V_4^-)$	$3(V_4^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$6(V_4^-)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(V_4^-)
$(\mathbb{Z}_2)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_4^-)$
$2(\mathbb{Z}_2^-)$	$(\mathbb{Z}_4^c)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_4^c)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_4^c)$
$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$
$2(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_2^-)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$
$(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$2(D_1^z)+(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$2(D_1^z)+5(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	(D_1^z)

Table A3.16. Multiplication table for the $A(S_4)$-module $A_1^t(S_4 \times S^1)$

	(A_5)	(A_4)	(D_5)	(D_3)	$(\mathbb{Z}_5)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
$(A_4^{t_1})$	$(A_4^{t_1})+(\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_3^t)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(V_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$(A_4^{t_1})$
$(A_4^{t_2})$	$(A_4^{t_2})+(\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_3^t)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(V_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$(A_4^{t_2})$
(D_5^z)	$(\mathbb{Z}_2^-)$	$(D_5^z)+(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$(\mathbb{Z}_6)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_5^z)
(D_3^z)	$(\mathbb{Z}_3)+(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$(D_3^z)+(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+3(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+4(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	(D_3^z)
$(\mathbb{Z}_5^{t_1})$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_1})+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_5^{t_2})$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_2})+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_2})+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_2})$
(V_4^-)	$(V_4^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+(\mathbb{Z}_2)$	$2(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(V_4^-)+3(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+6(\mathbb{Z}_1)$	$15(\mathbb{Z}_1)$	(V_4^-)
$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)+3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)+6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$20(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$
$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)+6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+14(\mathbb{Z}_1)$	$30(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table A3.17. Multiplication table for the $A(A_5)$-module $A_1^t(A_5 \times S^1)$

	$(O(2))$	$(SO(2))$	(D_m)
$(O(2))$	$(SO(2))$	(D_m)	$(O(2))$
$(SO(2))$	$2(SO(2))$	0	$(SO(2))$
(D_n)	0	$2(D_l)$, where $l = \gcd(n, m)$	(D_n)
$(O(2)^-)$	$(SO(2))$	(D_m^z)	$(O(2)^-)$
$(SO(2)^k)$	$2(SO(2)^k)$	0	$(SO(2)^k)$
(D_n^z)	0	$2(D_l^z)$, where $l = \gcd(n, m)$	(D_n^z)
(D_{2k}^d)	0	$\left\lvert\frac{k}{l}\right\rvert (D_l) + \left\lvert\frac{k}{l}\right\rvert (D_l^z) + 2\left(1 - \left\lvert\frac{k}{l}\right\rvert\right)(D_l^d)$ where $l = \gcd(m, 2k)$	(D_{2k}^d)

Table A3.18. Multiplication table for the $A(O(2))$-module $A_1^t(O(2) \times S^1)$

	$(\mathbb{Z}_m \times S^1)$	$(SO(2) \times \mathbb{Z}_{l_1})$	$(\mathbb{Z}_m \times \mathbb{Z}_{l_1})$	$(SO(2)^{\varphi_m, l_1})$	$(\mathbb{Z}_m^{\varphi_k, l_1})$
$(\mathbb{Z}_n \times S^1)$	0	$(\mathbb{Z}_n \times \mathbb{Z}_{l_1})$	0	$(\mathbb{Z}_n^{\varphi_m, l_1})$	0
$(SO(2) \times \mathbb{Z}_{l_2})$	$(\mathbb{Z}_m \times \mathbb{Z}_{l_2})$	0	0	$(\mathbb{Z}_m \times \mathbb{Z}_l)$	0
$(\mathbb{Z}_n \times \mathbb{Z}_{l_2})$	0	0	0	0	0
$(SO(2)^{\varphi_n, l_2})$	$(\mathbb{Z}_m^{\varphi_n, l_2})$	$(\mathbb{Z}_n \times \mathbb{Z}_l)$	0	$(\mathbb{Z}_{m-n}^{\varphi_n, l})$	0
$(SO(2)^{\varphi_m, l_2})$	$(\mathbb{Z}_m \times \mathbb{Z}_{l_2})$	$(\mathbb{Z}_m \times \mathbb{Z}_l)$	0	$(\mathbb{Z}_{2m}^{d, l})$	0
$(\mathbb{Z}_n^{\varphi_{k'}, l_2})$	0	0	0	0	0

where $l = \gcd(l_1, l_2)$ and $m > n$.

Table A3.19. Multiplication Table for the $U(T^2)$

	$(SO(2) \times S^1)$	$(D_m \times S^1)$	$(\mathbb{Z}_m \times S^1)$
$(SO(2) \times S^1)$	$2(SO(2) \times S^1)$	$(\mathbb{Z}_m \times S^1)$	$2(\mathbb{Z}_m \times S^1)$
$(D_n \times S^1)$	$(\mathbb{Z}_n \times S^1)$	$\begin{cases} 2(D_k \times S^1) - (\mathbb{Z}_k \times S^1) \\ k = \gcd(m,n) \end{cases}$	$\begin{cases} (\mathbb{Z}_k \times S^1) \\ k = \gcd(m,n) \end{cases}$
$(\mathbb{Z}_n \times S^1)$	$2(\mathbb{Z}_n \times S^1)$	$\begin{cases} (\mathbb{Z}_k \times S^1) \\ k = \gcd(m,n) \end{cases}$	0
$(O(2) \times \mathbb{Z}_l)$	$(SO(2) \times \mathbb{Z}_l)$	$(D_m) \times \mathbb{Z}_l)$	$(\mathbb{Z}_m \times \mathbb{Z}_l)$
$(SO(2) \times \mathbb{Z}_l)$	$2(SO(2) \times \mathbb{Z}_l)$	$(\mathbb{Z}_m \times \mathbb{Z}_l)$	$2(\mathbb{Z}_m \times \mathbb{Z}_l)$
$(D_n \times \mathbb{Z}_l)$	$(\mathbb{Z}_n \times \mathbb{Z}_l)$	$\begin{cases} 2(D_k \times \mathbb{Z}_l) - (\mathbb{Z}_k \times \mathbb{Z}_l), \\ k = gcd(n,m) \end{cases}$	0
$(\mathbb{Z}_n \times \mathbb{Z}_l)$	$2(\mathbb{Z}_n \times \mathbb{Z}_l)$	0	0
$(O(2)^{-,l})$	$(SO(2) \times \mathbb{Z}_l$	$(D_m^{z,l})$	$(\mathbb{Z}_m \times \mathbb{Z}_l)$
$(SO(2)^{\varphi_k,l})$	$2(SO(2)^{\varphi_k,l})$	$(\mathbb{Z}_m^{\varphi_k,l})$	$2(\mathbb{Z}_m^{\varphi_k,l})$
$(D_n^{z,l})$	$(\mathbb{Z}_n \times \mathbb{Z}_l)$	$\begin{cases} 2(D_k^{z,l}) - (\mathbb{Z}_k \times \mathbb{Z}_l), \\ k = \gcd(m,n) \end{cases}$	0
$\begin{cases} (D_{2n}^{d,l}) \\ m \quad \text{even} \end{cases}$	$(\mathbb{Z}_{2k}^{d,l})$	$\begin{cases} 2(D_{2k}^{d,l}) - (\mathbb{Z}_{2k}^{d,l}), \\ k = \gcd(m,n) \end{cases}$	0
$\begin{cases} (D_{2n}^{d,l}) \\ m \quad \text{odd} \end{cases}$	$(\mathbb{Z}_{2k}^{d,l})$	$\begin{cases} (D_k \times \mathbb{Z}_l) + (D_k^{z,l}) - (\mathbb{Z}_k \times \mathbb{Z}_l), \\ k = \gcd(m,2n) \end{cases}$	0
$(\mathbb{Z}_n^{\varphi_k,l})$	$2(\mathbb{Z}_n^{\varphi_k,l})$	0	0
$(\mathbb{Z}_{2n}^{d,l})$	$2(\mathbb{Z}_{2n}^{d,l})$	0	0

	$(O(2) \times \mathbb{Z}_{l_2})$	$(SO(2) \times \mathbb{Z}_{l_2})$	$(O(2)^{-,l})$	$(SO(2)^{\varphi_m,l_2})$	$(SO(2)^{\varphi_n,l_2})$
$(SO(2)^{\varphi_n,l_1})$	$2(\mathbb{Z}_n \times \mathbb{Z}_l)$	$2(\mathbb{Z}_n \times \mathbb{Z}_l)$	$2(\mathbb{Z}_n \times \mathbb{Z}_l)$	$(\mathbb{Z}_{n-m}^{\varphi_n,l}) + (\mathbb{Z}_{n+m}^{\varphi_m,l})$	$2(\mathbb{Z}_{2n}^{d,l})$

where $l = \gcd(l_1, l_2)$. All other products (except for $(O(2) \times S^1)$, which is the unit element in $U(O(2) \times S^1)$) are zero.

Table A3.20. Multiplication Table for $U(O(2) \times S^1)$

A4

Tables of Computational Results

A4.1 Results for Section 6.3

A4.1.1 Hopf Bifurcation in a FDE-System with D_5-Symmetry

Consider the system (6.42) with the matrix C of the type

$$C = \begin{bmatrix} c\,d\,0\,0\,d \\ d\,c\,d\,0\,0 \\ 0\,d\,c\,d\,0 \\ 0\,0\,d\,c\,d \\ d\,0\,0\,d\,c \end{bmatrix}, \tag{A4.1}$$

which is symmetric with respect to the dihedral group $\Gamma = D_5$ acting on $V = \mathbb{R}^5$. Let $\rho := e^{i\frac{2\pi}{5}}$ be the generator of $\mathbb{Z}_5$ and κ be the operator of complex conjugation. Notice that ρ acts on a vector $x = (x^0, x^1, \dots, x^4)$ by sending the k-th coordinate of x to the $k+1$ (mod 5) coordinate and κ acts by reversing the order of the components of x.

We have the following isotypical decomposition of V (cf. [15, 5] for details)

$$V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where $\mathcal{V}_i$ are explained in Appendix A2.2.4.

The spectrum of C is given by

$$\sigma(C) = \left\{ \xi_0^0 = c + 2d, \xi_1^1 = c + 2d\frac{\sqrt{5}-1}{4}, \xi_2^2 = c - 2d\frac{\sqrt{5}+1}{4} \right\}.$$

The dominating orbit types in W are (D_5), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$ and (D_1^z) (cf. Appendix A2.1.2 for definitions).

Using the command

$$\omega(\alpha_o, \beta_o)_1 = \texttt{showdegree}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r, \mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r),$$

we obtain the results for the D_5-symmetric Hopf bifurcation problem of the system (6.42) and organize them in Tables A4.1.

ξ_o^j	$\varepsilon_0, \varepsilon_1, \varepsilon_2$	$\omega(\alpha_j, \beta_j)_1$	# Branches
ξ_o^0	000	(D_5)	1
ξ_o^0	100	$-(D_5)$	1
ξ_o^0	001	$(D_5) - 2(D_1) + (\mathbb{Z}_1)$	1
ξ_o^0	110	$-(D_5) + 2(D_1) - (\mathbb{Z}_1)$	1
ξ_o^0	011	(D_5)	1
ξ_o^0	111	$-(D_5)$	1
ξ_o^1	000	$(\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
ξ_o^1	100	$-(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
ξ_o^1	001	$(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
ξ_o^1	110	$-(\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
ξ_o^1	011	$(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
ξ_o^1	111	$-(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
ξ_o^2	000	$(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
ξ_o^2	100	$-(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
ξ_o^2	001	$(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
ξ_o^2	110	$-(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
ξ_o^2	011	$(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
ξ_o^2	111	$-(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8

Table A4.1. Equivariant classification of the Hopf bifurcation with D_5 symmetries

A4.1.2 Hopf Bifurcation in a FDE-System with S_4-Symmetry

Consider the system (6.42) with the matrix C of the type

$$C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}, \tag{A4.2}$$

which is symmetric with respect to the octahedral group $\Gamma = S_4$, where S_4 acts on the space $V := \mathbb{R}^8$ by permuting the coordinates of the vectors in the same way as the symmetrics of a cube in $\mathbb{R}^3$ permute its eight vertices. It can be verified that the representation V has the following S_4-isotypical decomposition (cf. [15, 5] for details)

$$V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4,$$

where $\mathcal{V}_i$ are explained in Appendix A2.2.6. The spectrum of C is given by

$$\sigma(C) = \{\xi_0^0 = c + 3d, \xi_1^1 = c - 3d, \xi_2^3 = c + d, \xi_3^4 = c - d\}.$$

The dominating orbit types in W are (S_4), (S_4^-), (D_4^d), (D_2^d), $(\mathbb{Z}_4^c) := (\mathbb{Z}_4^{t_1})$, $(\mathbb{Z}_3^t) := (\mathbb{Z}_3^{t_1})$, (D_4^z) and (D_3^z) (cf. Appendix A2.1.4 for definitions).

Using the command

$$\omega(\alpha_o, \beta_o)_1 = \texttt{showdegree}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r, \mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r),$$

we obtain the results for the S_4-symmetric Hopf bifurcation problem of the system (6.42) and organize them in Tables A4.2.

A4.2 Results for Section 7.3

A4.2.1 Hopf Bifurcation in a NFDE-System with D_4-Symmetry

Consider the system (7.34) with the matrix Q, P_1 and P_2 of the type

$$C = \begin{bmatrix} c\ d\ 0\ d \\ d\ c\ d\ 0 \\ 0\ d\ c\ d \\ d\ 0\ d\ c \end{bmatrix}, \tag{A4.3}$$

which is symmetric with respect to the dihedral group $\Gamma = D_4$ acting on $V = \mathbb{R}^4$. Let $\xi := e^{i\frac{\pi}{2}}$ be the generator of $\mathbb{Z}_4$ and κ be the operator of complex conjugation. Notice that ξ acts on a vector $x = (x^0, x^1, x^2, x^3)$ by sending the k-th coordinate of x to the $k+1$ (mod 4) coordinate and κ acts by reversing the order of the components of x.

We have the D_4-isotypical decompositions

$$V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_3, \quad V^c = \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \mathcal{U}_3,$$

thus $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\} = \{\varepsilon_0, \varepsilon_1, \varepsilon_3\}$, and there are three types of bifurcation points (α_o, β_o) correspondingly. Since getting the complete list of the bifurcation invariants $\omega(\lambda_o)_1$ for the system (7.34) is a simple task of applying the Maple© package for the group $\Gamma = D_4$ by

$$\omega(\lambda_o)_1 = \texttt{showdegree}(\varepsilon_0, \varepsilon_1, 0, \varepsilon_3, 0, \mathfrak{t}_0, \mathfrak{t}_1, 0, \mathfrak{t}_3, 0),$$

we present in Table A4.3 only some selected results for the group D_4.

ξ_o^j	$\varepsilon_0,\varepsilon_1,\varepsilon_3,\varepsilon_4$	$\omega(\alpha_j,\beta_j)_1$	# Branches
ξ_0^0	0000	(S_4)	1
ξ_0^0	1000	$-(S_4)$	1
ξ_0^0	0100	$(S_4)-(A_4)$	1
ξ_0^0	1010	$-(S_4)+2(D_3)+(D_2)-3(D_1)+(\mathbb{Z}_1)$	1
ξ_0^0	0101	$(S_4)-(A_4)-(\mathbb{Z}_4)+(\mathbb{Z}_3)-(D_1)+(\mathbb{Z}_1)$	1
ξ_0^0	1011	$-(S_4)+2(D_3)+(D_2)+(\mathbb{Z}_4)-(\mathbb{Z}_3)-2(D_1)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_0^0	0111	$-(S_4)-(A_4)-2(D_3)-(D_2)-(\mathbb{Z}_4)+(\mathbb{Z}_3)+2(D_1)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	1
ξ_0^0	1111	$(S_4)+(A_4)+2(D_3)+(D_2)+(\mathbb{Z}_4)-(\mathbb{Z}_3)-2(D_1)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_1^1	0000	(S_4^-)	1
ξ_1^1	1000	$-(S_4^-)$	1
ξ_1^1	0100	$(S_4)-(A_4)$	1
ξ_1^1	1010	$-(S_4^-)+2(D_3^z)+(D_2^z)-3(D_1^z)+(\mathbb{Z}_1)$	1
ξ_1^1	0101	$(S_4^-)-(A_4)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)-(D_1^z)+(\mathbb{Z}_2)$	1
ξ_1^1	1011	$-(S_4^-)+2(D_3^z)+(D_2^z)+(\mathbb{Z}_4^-)-(\mathbb{Z}_3)-2(D_1^z)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_1^1	0111	$(S_4^-)-(A_4)-2(D_3^z)-(D_2^z)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)+2(D_1^z)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	1
ξ_1^1	1111	$-(S_4^-)+(A_4)+2(D_3^z)+(D_2^z)+(\mathbb{Z}_4)-(\mathbb{Z}_3)-2(D_1^z)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_2^3	0000	$(D_4^d)+(D_3)+(D_2^d)+(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-(D_1)-(\mathbb{Z}_2^-)$	24
ξ_2^3	1000	$-(D_4^d)-(D_3)-(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_3^t)+(D_1)+(\mathbb{Z}_2^-)$	24
ξ_2^3	0100	$(D_4^d)+(D_3)+(D_2^d)+(\mathbb{Z}_4^c)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1)-(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_2^3	1010	$-(D_4^d)+(D_3)+(D_2^d)-(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-(D_1^z)-3(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_2^3	0101	$(D_4^d)+(D_3)+(D_2^d)+(D_2)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4^-)-(V_4^-)+(\mathbb{Z}_3^t)-(D_1^z)-3(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_2^3	1011	$-(D_4^d)+(D_3)+(D_2^d)+(D_2)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_2^3	0111	$(D_4^d)-(D_3)-(D_2^d)-(D_2)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4^-)-(V_4^-)-(\mathbb{Z}_3^t)+(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)$	24
ξ_2^3	1111	$-(D_4^d)+(D_3)+(D_2^d)+(D_2)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)+(V_4^-)+(\mathbb{Z}_3^t)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)$	24
ξ_3^4	0000	$(D_4^z)+(D_3^z)+(D_2^d)+(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-(D_1^z)-(\mathbb{Z}_2^-)$	24
ξ_3^4	1000	$-(D_4^z)-(D_3^z)-(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_3^t)+(D_1^z)+(\mathbb{Z}_2^-)$	24
ξ_3^4	0100	$(D_4^z)+(D_3^z)+(D_2^d)+(\mathbb{Z}_4^c)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_3^4	1010	$-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)-(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-3(D_1^z)-(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_3^4	0101	$(D_4^z)+(D_3^z)+(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4)-(V_4^-)+(\mathbb{Z}_3^t)-3(D_1^z)-(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_3^4	1011	$-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_3^4	0111	$(D_4^z)-(D_3^z)-(D_2^d)-(D_2^z)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4)-(V_4^-)-(\mathbb{Z}_3^t)+(D_1^z)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)$	24
ξ_3^4	1111	$-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4)+(V_4^-)+(\mathbb{Z}_3^t)-(D_1^z)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)$	24

Table A4.2. Equivariant classification of the Hopf bifurcation with S_4 symmetries

A4.2.2 Hopf Bifurcation in a NFDE-System with A_5-Symmetry

Consider the system (7.34) with the matrix Q, P_1 and P_2 of the type (A4.4)

E_j	$\varepsilon_0, \varepsilon_1, \varepsilon_3$	$\omega(\alpha_o, \beta_o)_1$	# Branches
E_0	0,1,1	$(D_4) - (\mathbb{Z}_4) - (D_1) - (\tilde{D}_1) + (\mathbb{Z}_1)$	1
E_0	1,1,0	$-(D_4) + (D_1) + (\tilde{D}_1) - (\mathbb{Z}_1)$	1
E_0	1,1,1	$-(D_4) + (\mathbb{Z}_4) + (D_1) + (\tilde{D}_1) - (\mathbb{Z}_1)$	1
E_1	0,0,1	$-(\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (\mathbb{Z}_2^-)$	6
E_1	0,1,0	$(\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (\mathbb{Z}_2^-) - (D_1^z) - (\tilde{D}_1^z) - (D_1) - (\tilde{D}_1) + 2(\mathbb{Z}_1)$	6
E_1	0,1,1	$-(\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (\mathbb{Z}_2^-) - (D_1^z) - (\tilde{D}_1^z) - (D_1) - (\tilde{D}_1) + 2(\mathbb{Z}_1)$	6
E_1	1,1,0	$-(\mathbb{Z}_4^t) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) + (\tilde{D}_1^z) + (D_1) + (\tilde{D}_1) - 2(\mathbb{Z}_1)$	6
E_1	1,1,1	$(\mathbb{Z}_4^t) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) + (\tilde{D}_1^z) + (D_1) + (\tilde{D}_1) - 2(\mathbb{Z}_1)$	6
E_3	0,1,1	$(D_4^d) - (\mathbb{Z}_4^d) - (\tilde{D}_1^z) - (D_1) + (\mathbb{Z}_1)$	2
E_3	1,0,1	$-(D_4^d) + (\mathbb{Z}_4^d)$	2
E_3	1,1,0	$-(D_4^d) + (\tilde{D}_1^z) + (D_1) - (\mathbb{Z}_1)$	2
E_3	1,1,1	$-(D_4^d) + (\mathbb{Z}_4^d) + (\tilde{D}_1^z) + (D_1) - (\mathbb{Z}_1)$	2

Table A4.3. Examples of the equivariant classification of the Hopf bifurcation with D_4 symmetries

$$C = \begin{bmatrix}
c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & d & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & 0 & d & c
\end{bmatrix}. \tag{A4.4}$$

We have the following A_5-isotypical decompositions

$$V = \mathcal{V}_0 \oplus [\mathcal{V}_1 \oplus \mathcal{V}_1] \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4,$$
$$V^c = \mathcal{U}_0 \oplus [\mathcal{U}_1 \oplus \mathcal{U}_1] \oplus \mathcal{U}_2 \oplus \mathcal{U}_3 \oplus \mathcal{U}_4,$$

thus $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_m}\} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, and there are five types of bifurcation points (α_o, β_o) correspondingly. A partial list of the bifurcation invariants

$\omega(\lambda_o)_1$ for the system (7.34) is presented in Table A4.4, which was established by using the Maple© package for the group $\Gamma = A_5$,

$$\omega(\lambda_o)_1 = \texttt{showdegree}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \mathrm{t}_0, \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \mathrm{t}_4).$$

E_j	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$	$\omega(\lambda_o)_1$	# Branches
E_0	10101	$-(A_5) + 2(D_5) + 2(D_3) - (\mathbb{Z}_5) - (\mathbb{Z}_3) - 4(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	1
E_0	11101	$-(A_5) + 2(A_4) + 2(D_5) - (\mathbb{Z}_5) - 2(\mathbb{Z}_3) - 3(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	1
E_1	0000	$(A_4) + (D_3^z) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2)$	55
E_1	00100	$(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	55
E_1	00110	$(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	55
E_1	10001	$-(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + (\mathbb{Z}_3^t) + 3(\mathbb{Z}_3)$ $+3(\mathbb{Z}_2^-) + 3(\mathbb{Z}_2) - 4(\mathbb{Z}_1)$	55
E_1	10101	$-(A_4) + (D_3^z) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	55
E_2	00000	$(A_4^{t_1}) + (A_4^{t_2}) + (D_5) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) - 2(\mathbb{Z}_2)$	50
E_2	00110	$(A_4^{t_1}) + (A_4^{t_2}) - (D_5) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_5) + (V_4^-)$ $+(\mathbb{Z}_3) + 2(\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	50
E_2	01010	$-(A_4^{t_1}) - (A_4^{t_2}) + (D_5) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) - (V_4^-) + (\mathbb{Z}_1)$	50
E_2	10100	$-(A_4^{t_1}) - (A_4^{t_2}) + (D_5) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + 4(\mathbb{Z}_3^t)$ $+2(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 3(\mathbb{Z}_1)$	50
E_3	00010	$(D_5^z) + (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 4(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 3(\mathbb{Z}_1)$	44
E_3	00100	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	44
E_3	01010	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	44
E_3	10011	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (V_4^-) - (\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-)$	44
E_3	10100	$(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_1}) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	44
E_3	11110	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) + (V_4^-) - (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	44
E_4	00010	$(D_5^z) + (D_3^z) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 4(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 3(\mathbb{Z}_1)$	44
E_4	00100	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	44
E_4	01010	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	44
E_4	10011	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-)$	44
E_4	10100	$(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	44
E_4	11110	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) + (V_4^-) - (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	44

Table A4.4. Examples of the equivariant classification of the Hopf bifurcation with A_5 symmetries

A4.3 Results for Section 8.3

A4.3.1 Hopf Bifurcation in a FPDE-System with D_3-Symmetry

We assume here that the matrix C is of type

$$C = \begin{bmatrix} c & d & d \\ d & c & d \\ d & d & c \end{bmatrix}$$

with $c = -3$ and $d = -1$. In this case we have $\sigma(C) = \{{}^0\xi_0^0 = -5, {}^1\xi_1^1 = -2\}$, $m(\xi_0) = m(\xi_1) = 1$. The bifurcation invariants $\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1$ in this case are listed in Table A4.5, which was established by using the Maple routines for the group $\Gamma = D_3$, in the following way:

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 = (-1)^\nu \texttt{showdegree[D3]}(\varepsilon_0, \varepsilon_1, 0, m_0(\zeta_k), m_1(\zeta_k), 0).$$

${}^j\xi_o$	$\varepsilon_0, \varepsilon_1$	$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1$	#
${}^0\xi_0$	00	$(-1)^\nu \Big((D_3)\Big)$	1
${}^0\xi_0$	01	$(-1)^\nu \Big((D_3) - (\mathbb{Z}_3)\Big)$	1
${}^0\xi_0$	10	$(-1)^{\nu+1} \Big((D_3)\Big)$	1
${}^0\xi_0$	11	$(-1)^{\nu+1} \Big((D_3) - (\mathbb{Z}_3)\Big)$	1
${}^1\xi_1$	00	$(-1)^\nu \Big((\mathbb{Z}_3^z) + (D_1^z) + (D_1) - (\mathbb{Z}_1)\Big)$	6
${}^1\xi_1$	01	$(-1)^\nu \Big((\mathbb{Z}_3^z) - (D_1^z) - (D_1) + (\mathbb{Z}_1)\Big)$	6
${}^1\xi_1$	10	$(-1)^{\nu+1} \Big((\mathbb{Z}_3^z) + (D_1^z) + (D_1) - (\mathbb{Z}_1)\Big)$	6
${}^1\xi_1$	11	$(-1)^{\nu+1} \Big((\mathbb{Z}_3^z) - (D_1^z) - (D_1) + (\mathbb{Z}_1)\Big)$	6

Table A4.5. Equivariant classification of the Hopf bifurcation with D_3 symmetries

A4.3.2 Hopf Bifurcation in a FPDE-System with A_4-Symmetry

We assume here that the matrix C is of type

$$C = \begin{bmatrix} c & d & d & d \\ d & c & d & d \\ d & d & c & d \\ d & d & d & c \end{bmatrix}$$

with $c = -4$ and $d = 1$. Clearly, C is A_4-equivariant. In this case we have $\sigma(C) = \{-1, -5\}$. We classify the eigenvalues of C as ${}^0\xi_0^0 = -1, {}^3\xi_1^3 = -5$

and we have the following multiplicities $m(\xi_0) = m(\xi_1) = 1$. Sample invariants $\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1$ in this case are listed in Table A4.6. To obtain the other invariants, use the Maple routines for the group $\Gamma = A_4$:

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 = (-1)^\nu \texttt{showdegree[A4]}(\varepsilon_0, 0, \varepsilon_3, m_0(\zeta_k), 0, 0, m_3(\zeta_k)).$$

${}^j\xi_o$	$\varepsilon_0, \varepsilon_3$	$\omega(\alpha_{\nu,m,k}; \beta_{\nu,m,k}, 0)_1$	#
${}^0\xi_0$	00	$(-1)^\nu\big((A_4)\big)$	1
${}^0\xi_0$	01	$(-1)^\nu\big((A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1)\big)$	1
${}^0\xi_0$	10	$(-1)^{\nu+1}\big((A_4)\big)$	1
${}^0\xi_0$	11	$(-1)^{\nu+1}\big((A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1)\big)$	1
${}^3\xi_1$	00	$(-1)^\nu\big((V_4^-) + (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (\mathbb{Z}_3) - (\mathbb{Z}_1)\big)$	12
${}^3\xi_1$	01	$(-1)^\nu\big((V_4^-) - (\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)\big)$	12
${}^3\xi_1$	10	$(-1)^{\nu+1}\big((V_4^-) + (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (\mathbb{Z}_3) - (\mathbb{Z}_1)\big)$	12
${}^3\xi_1$	11	$(-1)^{\nu+1}\big((V_4^-) - (\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)\big)$	12

Table A4.6. Equivariant classification of the Hopf bifurcation with A_4 symmetries

A4.4 Results for Section 9.3

A4.4.1 Existence in Q_8-Symmetric Lotka-Volterra Type System

The quaternionic group Q_8 can be described as a subgroup of S_8 generated by

$$i := (1324)(5867), \quad j := (1526)(3748).$$

We consider the space $V := \mathbb{R}^8$ on which Q_8 acts by permuting the coordinates of vectors $x \in V$. Consider the matrix

$$A := \begin{bmatrix} a & c & b & b & d & d & e & e \\ c & a & b & b & d & d & e & e \\ b & b & a & c & e & e & d & d \\ b & b & c & a & e & e & d & d \\ d & d & e & e & a & c & b & b \\ d & d & e & e & c & a & b & b \\ e & e & d & d & b & b & a & c \\ e & e & d & d & b & b & c & a \end{bmatrix}$$

The matrix A commutes with the Q_8-action on V. The matrix A has the following eigenvalues and eigenspaces:

$$\begin{aligned}
\mu_1 &:= a - 2e + c - 2b + 2d, && \widetilde{E}(\mu_1) \simeq \mathcal{V}_2\\
\mu_2 &:= a - 2e + c + 2b - 2d, && \widetilde{E}(\mu_2) \simeq \mathcal{V}_1,\\
\mu_3 &:= a - c, && \widetilde{E}(\mu_3) \simeq \mathcal{V}_4 \text{ (quaternionic type)},\\
\mu_4 &:= a + 2e + c - 2b - 2d, && \widetilde{E}(\mu_4) \simeq \mathcal{V}_3,\\
\mu_5 &:= a + 2e + c + 2b + 2d, && \widetilde{E}(\mu_5) \simeq \mathcal{V}_0.
\end{aligned}$$

For definiteness, we choose the positive entries of A being $a = 8$, $b = 1$, $c = 3$, $d = 2$, $e = 1.5$ and $\tau = 4$, so

$$\tau\mu_1 = 40,\quad \tau\mu_2 = 24,\quad \tau\mu_3 = 20,\quad \tau\mu_4 = 32,\quad \tau\mu_5 = 80,$$

so we can easily determine (from Table 9.1 the values $n(\mu_i)$, i.e.

$$n(\mu_1) = 6,\quad n(\mu_2) = 3,\quad n(\mu_3) = 2,\quad n(\mu_4) = 4,\quad n(\mu_5) = 12.$$

Then we have

$$\begin{gathered}
\mathfrak{m}_{0,1} = 1,\quad \mathfrak{m}_{0,2} = 2,\quad \mathfrak{m}_{0,3} = 3,\quad \mathfrak{m}_{0,4} = 4,\quad \mathfrak{m}_{0,5} = 5,\quad \mathfrak{m}_{0,6} = 6,\\
\mathfrak{m}_{0,7} = 7,\quad \mathfrak{m}_{0,8} = 5,\quad \mathfrak{m}_{0,9} = 4,\quad \mathfrak{m}_{0,10} = 3,\quad \mathfrak{m}_{0,11} = 2,\quad \mathfrak{m}_{0,12} = 1,\\
\mathfrak{m}_{1,1} = 1,\quad \mathfrak{m}_{1,2} = 2,\quad \mathfrak{m}_{1,3} = 1,\quad \mathfrak{m}_{2,1} = 1,\quad \mathfrak{m}_{2,2} = 2,\quad \mathfrak{m}_{2,3} = 3,\\
\mathfrak{m}_{2,4} = 3,\quad \mathfrak{m}_{2,5} = 2,\quad \mathfrak{m}_{2,6} = 1,\quad \mathfrak{m}_{3,1} = 1,\quad \mathfrak{m}_{3,2} = 2,\quad \mathfrak{m}_{3,3} = 2,\\
\mathfrak{m}_{3,4} = 1,\quad \mathfrak{m}_{4,1} = 2,\quad \mathfrak{m}_{4,1} = 2.
\end{gathered}$$

By applying formula (9.41) we obtain

$$\begin{aligned}
\boxdot = {} & \deg_{\mathcal{V}_{0,1}} + 2\deg_{\mathcal{V}_{0,2}} + 3\deg_{\mathcal{V}_{0,3}} + 4\deg_{\mathcal{V}_{0,4}} + 5\deg_{\mathcal{V}_{0,5}} + 6\deg_{\mathcal{V}_{0,6}} + 7\deg_{\mathcal{V}_{0,7}}\\
& + 5\deg_{\mathcal{V}_{0,8}} + 4\deg_{\mathcal{V}_{0,9}} + 3\deg_{\mathcal{V}_{0.10}} + 2\deg_{\mathcal{V}_{0,11}} + \deg_{\mathcal{V}_{0.12}} + \deg_{\mathcal{V}_{1.1}} + 2\deg_{\mathcal{V}_{1,2}}\\
& + \deg_{\mathcal{V}_{1,3}} + \deg_{\mathcal{V}_{2,1}} + 2\deg_{\mathcal{V}_{2,2}} + 3\deg_{\mathcal{V}_{2,3}} + 3\deg_{\mathcal{V}_{2,4}} + 2\deg_{\mathcal{V}_{2,5}} + \deg_{\mathcal{V}_{2,6}}\\
& + \deg_{\mathcal{V}_{3,1}} + 2\deg_{\mathcal{V}_{3,2}} + 2\deg_{\mathcal{V}_{3,3}} + \deg_{\mathcal{V}_{3,4}} + 2\deg_{\mathcal{V}_{4,1}} + 2\deg_{\mathcal{V}_{4,1}},
\end{aligned}$$

where

$$\begin{aligned}
&\deg_{\mathcal{V}_{0,1}} = (\mathbf{Q_8}),\quad \deg_{\mathcal{V}_{k,1}} = (\mathbf{Q_8^{k-}}),\quad k = 1,2,3\\
&\deg_{\mathcal{V}_{4,1}} = (\mathbb{Z}_4^{1+}) + (\mathbb{Z}_4^{2+}) + (\mathbb{Z}_4^{3+}) + (\mathbb{Z}_4^{4+}) + -(\mathbb{Z}_2^{-})
\end{aligned}$$

The dominating orbit types in $\mathbb{H}^*$ are (Q_8), (Q_8^{k-}) and $(\mathbb{Z}_4^{k+})$ for $k = 1,2,3$. Consequently, we obtain

- there is at least 1 nonstationary periodic solution with symmetry (Q_8) ,
- there is at least 1 nonstationary periodic solution with symmetry (Q_8^{1-}),
- there are at least 1 nonstationary periodic solution with symmetries (Q_8^{2-}),
- there are at least 1 nonstationary periodic solution with symmetries (Q_8^{3-}),
- there are at least 2 nonstationary periodic solutions with symmetries $(\mathbb{Z}_4^{1+})$,
- there are at least 2 nonstationary periodic solutions with symmetries $(\mathbb{Z}_4^{2+})$,
- there are at least 2 nonstationary periodic solutions with symmetries $(\mathbb{Z}_4^{3+})$.

In summary, there exist at least 10 nonconstant periodic solutions of (9.20).

A4.4.2 Existence in D_8-Symmetric Lotka-Volterra Type System

Consider the dihedral group $D_8 = \{1, \gamma, \gamma^2, \dots, \gamma^7, \kappa, \kappa\gamma, \gamma^2, \dots, \kappa\gamma^7\} \subset O(2)$, where γ can be identified with $e^{\frac{\pi i}{4}}$ (i.e. γ is a complex linear operator $\gamma(z) = e^{\frac{\pi i}{4}} z$) and $\kappa := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We consider the space $V := \mathbb{R}^8$, where $\gamma \in D_8$ acts on a vector $(x_1, x_2, \dots, x_8)$ by sending the k-th coordinate of x to the $k+1$ (mod n) coordinate and $\kappa \in D_8$ acts by reversing the order of the components of x. Consider the following D_8-equivariant matrix A

$$A := \begin{bmatrix} d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}$$

The matrix A has the following eigenvalues and the corresponding eigenspaces

$$\begin{aligned} \mu_1 &:= c + 2d, & \widetilde{E}(\mu_1) &\simeq \mathcal{V}_0 \\ \mu_2 &:= c + \sqrt{2}d, & \widetilde{E}(\mu_2) &\simeq \mathcal{V}_1, \\ \mu_3 &:= c, & \widetilde{E}(\mu_3) &\simeq \mathcal{V}_2, \\ \mu_4 &:= c - \sqrt{2}d, & \widetilde{E}(\mu_4) &\simeq \mathcal{V}_3, \\ \mu_5 &:= c - 2d, & \widetilde{E}(\mu_5) &\simeq \mathcal{V}_5. \end{aligned}$$

For definiteness, we choose the positive entries of A by letting $c = 9$, $d = 3$ and $\tau = 4$, so

$$\tau\mu_1 = 60, \quad \tau\mu_2 \approx 52.97, \quad \tau\mu_3 = 36, \quad \tau\mu_4 \approx 19.03, \quad \tau\mu_5 = 12.$$

To determine the numbers $n(\mu_i)$, for $i = 0, 1, 2, 3, 5$, we list the approximate values of $\frac{\pi}{2} + 2n\pi$ we use Table 9.1 so, we have

$$n(\mu_1) = 9, \quad n(\mu_2) = 8, \quad n(\mu_3) = 5, \quad n(\mu_4) = 2, \quad n(\mu_5) = 1.$$

Let $j_1 = 2$ and $j_2 = 1$. Then,

$$\begin{gathered}
\mathfrak{m}_{0,1} = 1, \quad \mathfrak{m}_{0,2} = 2, \quad \mathfrak{m}_{0,3} = 3, \quad \mathfrak{m}_{0,4} = 4, \quad \mathfrak{m}_{0,5} = 5, \quad \mathfrak{m}_{0,6} = 4, \\
\mathfrak{m}_{0,7} = 3, \quad \mathfrak{m}_{0,8} = 2, \quad \mathfrak{m}_{0,9} = 1, \quad \mathfrak{m}_{1,1} = 1, \quad \mathfrak{m}_{1,2} = 2, \quad \mathfrak{m}_{1,3} = 3, \\
\mathfrak{m}_{1,3} = 4, \quad \mathfrak{m}_{1,5} = 4, \quad \mathfrak{m}_{1,6} = 3, \quad \mathfrak{m}_{1,7} = 2, \quad \mathfrak{m}_{1,8} = 1, \quad \mathfrak{m}_{2,1} = 1, \\
\mathfrak{m}_{2,2} = 2, \quad \mathfrak{m}_{2,3} = 3, \quad \mathfrak{m}_{2,4} = 2, \quad \mathfrak{m}_{2,5} = 1, \quad \mathfrak{m}_{3,1} = 1, \quad \mathfrak{m}_{3,2} = 2, \\
\mathfrak{m}_{3,3} = 1, \quad \mathfrak{m}_{5,1} = 1.
\end{gathered}$$

By applying formula (9.41) we obtain

$$\begin{aligned}
\boxdot = {} & \deg_{\mathcal{V}_{0,1}} + 2\deg_{\mathcal{V}_{0,2}} + 3\deg_{\mathcal{V}_{0,3}} + 4\deg_{\mathcal{V}_{0,4}} + 5\deg_{\mathcal{V}_{0,5}} + 4\deg_{\mathcal{V}_{0,6}} + 3\deg_{\mathcal{V}_{0,7}} \\
& + 2\deg_{\mathcal{V}_{0,8}} + \deg_{\mathcal{V}_{0,9}} + \deg_{\mathcal{V}_{1,1}} + 2\deg_{\mathcal{V}_{1,2}} + 3\deg_{\mathcal{V}_{1,3}} + 4\deg_{\mathcal{V}_{1,4}} + 4\deg_{\mathcal{V}_{1,5}} \\
& + 3\deg_{\mathcal{V}_{1,6}} + 2\deg_{\mathcal{V}_{1,7}} + \deg_{\mathcal{V}_{1,8}} + \deg_{\mathcal{V}_{2,1}} + 2\deg_{\mathcal{V}_{2,2}} + 3\deg_{\mathcal{V}_{2,3}} + 2\deg_{\mathcal{V}_{2,4}} \\
& + \deg_{\mathcal{V}_{2,5}} + \deg_{\mathcal{V}_{3,1}} + 2\deg_{\mathcal{V}_{3,2}} + \deg_{\mathcal{V}_{3,3}} + \deg_{\mathcal{V}_{5,1}},
\end{aligned}$$

where

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (\mathbf{D_8}), \\
\deg_{\mathcal{V}_{1,1}} &= (\mathbb{Z}_8^{\mathbf{t_1}}) + (\widetilde{D}_2^d) + (D_2^d) - (\mathbb{Z}_2^-), \\
\deg_{\mathcal{V}_{2,1}} &= (\widetilde{\mathbf{D}}_4^{\mathbf{d}}) + (D_4^d) + (\mathbb{Z}_8^{\mathbf{t_2}}) - (\mathbb{Z}_4^d), \\
\deg_{\mathcal{V}_{3,1}} &= (\mathbb{Z}_8^{\mathbf{t_3}}) + (\widetilde{D}_2^d) + (D_2^d) - (\mathbb{Z}_2^-), \\
\deg_{\mathcal{V}_{5,1}} &= (\mathbf{D_8^d}).
\end{aligned}$$

The dominating orbit types in $\mathbb{H}^*$ are (D_8), (D_8^d), $(\mathbb{Z}_8^{t_1})$, $(\mathbb{Z}_8^{t_2})$, $(\mathbb{Z}_8^{t_3})$ and $(\widetilde{D}_4^d)$. Consequently, we obtain

- there is at least 1 nonstationary periodic solution with symmetry (D_8),

- there is at least 1 nonstationary periodic solution with symmetry (D_8^d),
- there are at least 2 nonstationary periodic solutions with symmetries (D_4^d),
- there are at least 2 nonstationary periodic solutions with symmetries $(\mathbb{Z}_8^{t_1})$,
- there are at least2 nonstationary periodic solutions with least symmetries $(\mathbb{Z}_8^{t_2})$,
- there are at least 2 nonstationary periodic solutions with least symmetries $(\mathbb{Z}_8^{t_3})$.

In summary, there exist at least 10 nonconstant periodic solutions of (9.20).

A4.4.3 Existence in S_4-Symmetric Lotka-Volterra Type System

Assume that the octahedral group S_4 acts on $V := \mathbb{R}^8$ by permuting the coordinates in such a way that $(1234) \in S_4$ corresponds to the permutation $(1234)(5678) \in S_8$ and $(12) \in S_4$ corresponds to $(17)(28)34)(56) \in S_8$ (i.e. S_4 acts on V in the same way as it permutes the vertices of a regular cube). Consider the matrix

$$A := \begin{bmatrix} a & b & c & b & b & c & d & c \\ b & a & b & c & c & b & c & d \\ c & b & a & b & d & c & b & c \\ b & c & b & a & c & d & c & b \\ b & c & d & c & a & b & c & b \\ c & b & c & d & b & a & b & c \\ d & c & b & c & c & b & a & b \\ c & d & c & b & b & c & b & a \end{bmatrix}.$$

The matrix A commutes with the S_4-action on V. The matrix A has the following eigenvalues and eigenspaces:

$$\begin{aligned} \mu_1 &:= 3b + a + 3c + d, & \widetilde{E}(\mu_1) &\simeq \mathcal{V}_0 \\ \mu_2 &:= -3b + a + 3c - d, & \widetilde{E}(\mu_2) &\simeq \mathcal{V}_1, \\ \mu_3 &:= -b + a - c + d, & \widetilde{E}(\mu_3) &\simeq \mathcal{V}_4, \\ \mu_4 &:= b + a - c - d, & \widetilde{E}(\mu_4) &\simeq \mathcal{V}_3. \end{aligned}$$

For definiteness, we choose the positive entries of A being $a = 6$, $b = 1$, $c = 2$, $d = 2.5$ and $\tau = 4$, so

$$\tau\mu_1 = 70, \quad \tau\mu_2 = 26, \quad \tau\mu_3 = 22, \quad \tau\mu_4 = 10,$$

so we can easily determine (from Table 9.1 the values $n(\mu_i)$, i.e.

$$n(\mu_1) = 10, \ \ n(\mu_2) = 3, \ \ n(\mu_3) = 3, \ \ n(\mu_4) = 1.$$

As before, we choose $j_2 = 1$ and $j_1 = 2$. Then we have

$$\begin{gathered}
\mathfrak{m}_{0,1} = 1, \ \ \mathfrak{m}_{0,2} = 2, \ \ \mathfrak{m}_{0,3} = 3, \ \ \mathfrak{m}_{0,4} = 4, \ \ \mathfrak{m}_{0,5} = 5, \ \ \mathfrak{m}_{0,6} = 5, \\
\mathfrak{m}_{0,7} = 4, \ \ \mathfrak{m}_{0,8} = 3, \ \ \mathfrak{m}_{0,9} = 2, \ \ \mathfrak{m}_{0,10} = 1, \ \ \mathfrak{m}_{0,11} = 2, \ \ \mathfrak{m}_{0,12} = 1, \\
\mathfrak{m}_{1,1} = 1, \ \ \mathfrak{m}_{1,2} = 2, \ \ \mathfrak{m}_{1,3} = 1, \ \ \mathfrak{m}_{3,1} = 1, \ \ \mathfrak{m}_{3,2} = 2, \ \ \mathfrak{m}_{3,3} = 1, \\
\mathfrak{m}_{4,1} = 1.
\end{gathered}$$

By applying formula (9.41) we obtain

$$\begin{aligned}
\boxdot = {} & \deg_{\mathcal{V}_{0,1}} + 2\deg_{\mathcal{V}_{0,2}} + 3\deg_{\mathcal{V}_{0,3}} + 4\deg_{\mathcal{V}_{0,4}} + 5\deg_{\mathcal{V}_{0,5}} + 5\deg_{\mathcal{V}_{0,6}} + 4\deg_{\mathcal{V}_{0,7}} \\
& + 3\deg_{\mathcal{V}_{0,8}} + 2\deg_{\mathcal{V}_{0,9}} + \deg_{\mathcal{V}_{0,10}} + \deg_{\mathcal{V}_{1,1}} + 2\deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{1,3}} + \deg_{\mathcal{V}_{3,1}} \\
& + 2\deg_{\mathcal{V}_{3,2}} + \deg_{\mathcal{V}_{3,3}} + \deg_{\mathcal{V}_{4,1}},
\end{aligned}$$

where

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (\mathbf{S_4}), \\
\deg_{\mathcal{V}_{1,1}} &= (\mathbf{S_4^-}), \\
\deg_{\mathcal{V}_{3,1}} &= (\mathbb{Z}_4^{\mathbf{c}}) + (\mathbf{D_4^d}) + (\mathbf{D_2^d}) + (D_3) + (\mathbb{Z}_3^{\mathbf{t}}) - (\mathbb{Z}_2^-) - (D_1), \\
\deg_{\mathcal{V}_{4,1}} &= (\mathbb{Z}_4^{\mathbf{c}}) + (\mathbf{D_4^z}) + (D_2^d) + (D_3^z) + (\mathbb{Z}_3^{\mathbf{t}}) - (\mathbb{Z}_2^-) - (D_1^z).
\end{aligned}$$

The dominating orbit types in $\mathbb{H}^*$ are (S_4), (S_4^-), (D_4^d), (D_2^d), $(\mathbb{Z}_4^c)$, $(\mathbb{Z}_3^t)$ and (D_4^z). Consequently, we obtain

- there is at least 1 nonstationary periodic solution with symmetry (S_4),
- there is at least 1 nonstationary periodic solution with symmetry (S_4^-),
- there are at least 3 nonstationary periodic solutions with symmetries (D_4^d),
- there are at least 6 nonstationary periodic solutions with symmetries (D_2^d),
- there are at least 6 nonstationary periodic solutions with least symmetries $(\mathbb{Z}_4^c)$,
- there are at least 8 nonstationary periodic solutions with least symmetries $(\mathbb{Z}_3^t)$,
- there are at least 3 nonstationary periodic solutions with least symmetries (D_4^z).

In summary, there exist at least 28 nonconstant periodic solutions of (9.20).

Remark A4.4.1. One can consider other symmetry groups in (9.20), such as D_3, D_4, D_5, D_6, D_7, D_9, D_{10}, D_{11}, D_{12}, A_4 or A_5, for which there exists already developed computational database (including Maple© routines for the twisted equivariant degree). As it is clear from the formula (9.41) and the above examples, the similar existence results for all these groups can be easily obtained.

A4.5 Results for Section 10.1

Consider the system (10.1) assuming that (A1)–(A5). As the symmetry group Γ, take the dihedral groups D_4, D_5, D_6, the octahedral group S_4 and the icosahedral group A_5. Assume that $V := \mathbb{R}^n$ is an orthogonal Γ-representation, where Γ acts on $u = (u_1, u_2, \dots, u_n) \in V$ by permuting its coordinates. Moreover, for $\Gamma = D_n$, assume that C is of the type

$$C = \begin{bmatrix} c\, d\, 0 \dots 0\, d \\ d\, c\, d \dots 0\, 0 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots\ \vdots \\ d\, 0\, 0 \dots d\, c \end{bmatrix}.$$

For $\Gamma = S_4$, C is of the type

$$C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}.$$

For $\Gamma = A_5$, C is of the type

$$
C=\begin{bmatrix}
c&d&0&0&d&0&0&d&0&0&0&0&0&0&0&0&0&0&0&0\\
d&c&d&0&0&0&0&0&0&d&0&0&0&0&0&0&0&0&0&0\\
0&d&c&d&0&0&0&0&0&0&0&d&0&0&0&0&0&0&0&0\\
0&0&d&c&d&0&0&0&0&0&0&0&0&d&0&0&0&0&0&0\\
d&0&0&d&c&d&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&d&c&d&0&0&0&0&0&0&0&d&0&0&0&0&0\\
0&0&0&0&0&d&c&d&0&0&0&0&0&0&0&0&d&0&0&0\\
d&0&0&0&0&0&d&c&d&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&d&c&d&0&0&0&0&0&0&0&d&0&0\\
0&d&0&0&0&0&0&0&d&c&d&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&d&c&d&0&0&0&0&0&0&d&0\\
0&0&d&0&0&0&0&0&0&0&d&c&d&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&d&c&d&0&0&0&0&0&d\\
0&0&0&d&0&0&0&0&0&0&0&0&d&c&d&0&0&0&0&0\\
0&0&0&0&0&d&0&0&0&0&0&0&0&d&c&d&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&d&c&d&0&0&d\\
0&0&0&0&0&0&d&0&0&0&0&0&0&0&0&d&c&d&0&0\\
0&0&0&0&0&0&0&0&d&0&0&0&0&0&0&0&d&c&d&0\\
0&0&0&0&0&0&0&0&0&0&d&0&0&0&0&0&0&d&c&d\\
0&0&0&0&0&0&0&0&0&0&0&0&d&0&0&d&0&0&d&c
\end{bmatrix}.
$$

For definiteness, let $c = 4.5$, $d = 1$ for the matrix A, and $c = 9.5$, $d = 1$ for the matrix B.

A4.5.1 Existence in D_4-Symmetric Auto. Newtonian System

In the case $\Gamma = D_4$, we have $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_3$, to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_3) = (1, 1, 1)$ and $\sigma(A) = \{\xi_0^0 = 6.5, \xi_1^1 = 4.5, \xi_3^3 = 2.5\}$, $\sigma(B) = \{\xi_0^0 = 11.5, \xi_1^1 = 9.5, \xi_3^3 = 7.5\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$
\begin{aligned}
\widetilde{m}_0^2(A) &= 1, & \widetilde{m}_0^3(B) &= 1,\\
\widetilde{m}_1^2(A) &= 1, & \widetilde{m}_1^3(B) &= 1,\\
\widetilde{m}_3^1(A) &= 1, & \widetilde{m}_3^2(B) &= 1.
\end{aligned}
$$

Consequently, we have the non-zero isotypical defect numbers

$$
\mathfrak{m}_0^2 = -1, \quad \mathfrak{m}_0^3 = 1, \quad \mathfrak{m}_1^2 = -1, \quad \mathfrak{m}_1^3 = 1, \quad \mathfrak{m}_3^1 = -1, \quad \mathfrak{m}_3^2 = 1.
$$

Hence,

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=1}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= (-1)\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\right)\\
&+(-1)\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{3,1}}+1\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}\right)\\
&=\deg_{\mathcal{V}_{0,3}}+\deg_{\mathcal{V}_{1,3}}+\deg_{\mathcal{V}_{3,2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\deg{}^{t} = {}& \Theta_3\left[\texttt{showdegree[D4]}\,(1,1,0,1,0,1,1,0,0,0)\right]\\
&+\Theta_2\left[\texttt{showdegree[D4]}\,(1,1,0,1,0,0,0,0,1,0)\right].
\end{aligned}
$$

The dominating orbit types in W are (D_4), $(\mathbb{Z}_4^t) := (\mathbb{Z}_4^{t_1})$, (D_2^d), $(\widetilde{D}_2^d)$ and (D_4^d). The value of $\deg{}^{t}$ is listed in Table A4.7.

A4.5.2 Existence in D_5-Symmetric Auto. Newtonian System

In the case $\Gamma = D_5$, we have $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2$, to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_2) = (1,1,1)$ and $\sigma(A) = \{\xi_0^0 = 6.5, \xi_1^1 = 4.5 + \frac{\sqrt{5}-1}{2}, \xi_2^2 = 4.5 - \frac{\sqrt{5}+1}{2}\}$, $\sigma(B) = \{\xi_0^0 = 11.5, \xi_1^1 = 9.5 + \frac{\sqrt{5}-1}{2}, \xi_2^2 = 9.5 - \frac{\sqrt{5}+1}{2}\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$
\begin{aligned}
\widetilde{m}_0^2(A) &= 1, & \widetilde{m}_0^3(B) &= 1,\\
\widetilde{m}_1^2(A) &= 1, & \widetilde{m}_1^3(B) &= 1,\\
\widetilde{m}_2^1(A) &= 1, & \widetilde{m}_2^2(B) &= 1.
\end{aligned}
$$

Consequently, we have the non-zero isotypical defect numbers

$$
\mathfrak{m}_0^2 = -1,\quad \mathfrak{m}_0^3 = 1,\quad \mathfrak{m}_1^2 = -1,\quad \mathfrak{m}_1^3 = 1,\quad \mathfrak{m}_2^1 = -1,\quad \mathfrak{m}_2^2 = 1.
$$

Hence,

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=1}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= (-1)\cdot\left(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{0,2}}\right) + 1\cdot\left(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{0,3}}\right)\\
&+ (-1)\cdot\left(\deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{1,2}}\right) + 1\cdot\left(\deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{1,3}}\right)\\
&+ (-1)\cdot\deg_{\mathcal{V}_{2,1}} + 1\cdot\left(\deg_{\mathcal{V}_{2,1}} + \deg_{\mathcal{V}_{2,2}}\right)\\
&= \deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}} + \deg_{\mathcal{V}_{2,2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\deg^{t} &= \Theta_3\left[\texttt{showdegree[D5]}\,(1,1,1,0,1,1,0,0)\right]\\
&\quad + \Theta_2\left[\texttt{showdegree[D5]}\,(1,1,1,0,0,0,1,0)\right].
\end{aligned}
$$

The dominating orbit types in W are (D_5), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$ and (D_1^z). The value of $\deg^{t}$ is listed in Table A4.7.

A4.5.3 Existence in D_6-Symmetric Auto. Newtonian System

In the case $\Gamma = D_6$, we have $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4$, to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_4) = (1,1,1,1)$ and $\sigma(A) = \{\xi_0^0 = 6.5, \xi_1^1 = 5.5, \xi_2^2 = 3.5, \xi_4^4 = 2.5\}$, $\sigma(B) = \{\xi_0^0 = 11.5, \xi_1^1 = 10.5, \xi_2^2 = 8.5, \xi_4^4 = 7.5\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$
\begin{aligned}
\widetilde{m}_0^2(A) &= 1, & \widetilde{m}_0^3(B) &= 1,\\
\widetilde{m}_1^2(A) &= 1, & \widetilde{m}_1^3(B) &= 1,\\
\widetilde{m}_2^1(A) &= 1, & \widetilde{m}_2^2(B) &= 1,\\
\widetilde{m}_4^1(A) &= 1, & \widetilde{m}_4^2(B) &= 1.
\end{aligned}
$$

Consequently, we have the non-zero isotypical defect numbers

$$
\begin{aligned}
&\mathfrak{m}_0^2 = -1, \quad \mathfrak{m}_0^3 = 1, \quad \mathfrak{m}_1^2 = -1, \quad \mathfrak{m}_1^3 = 1,\\
&\mathfrak{m}_2^1 = -1, \quad \mathfrak{m}_2^2 = 1, \quad \mathfrak{m}_4^1 = -1, \quad \mathfrak{m}_4^2 = 1.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=1}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&=(-1)\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\right)\\
&+(-1)\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{2,1}}+(-1)\cdot\left(\deg_{\mathcal{V}_{2,1}}+\deg_{\mathcal{V}_{2,2}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{4,1}}+(-1)\cdot\left(\deg_{\mathcal{V}_{4,1}}+\deg_{\mathcal{V}_{4,2}}\right)\\
&=\deg_{\mathcal{V}_{0,3}}+\deg_{\mathcal{V}_{1,3}}+\deg_{\mathcal{V}_{2,2}}+\deg_{\mathcal{V}_{4,2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\deg^{\,t}=\;&\Theta_3\left[\texttt{showdegree[D6]}\,(1,1,1,0,1,0,1,1,0,0,0,0)\right]\\
&+\Theta_2\left[\texttt{showdegree[D6]}\,(1,1,1,0,1,0,0,0,1,0,1,0)\right].
\end{aligned}
$$

The dominating orbit types in W are (D_6), (D_6^d), $(\mathbb{Z}_6^{t_1})$, $(\mathbb{Z}_6^{t_2})$, (D_2^d) and (D_2^z). The value of $\deg^{\,t}$ is listed in Table A4.7.

A4.5.4 Existence in S_4-Symmetric Auto. Newtonian System

For the octahedral group S_4 we consider the representation $V=\mathbb{R}^8$, which has the isotypical decomposition $V=\mathcal{V}_0\oplus\mathcal{V}_1\oplus\mathcal{V}_3\oplus\mathcal{V}_4$, to which we associate the sequence $(\varepsilon_0,\varepsilon_1,\varepsilon_3,\varepsilon_4)=(1,1,1,1)$, and $\sigma(A)=\{\xi_0^0=7.5,\xi_1^1=1.5,\xi_2^3=5.5,\zeta_3^4=3.5\}$, $\sigma(B)=\{\xi_0^0=12.5,\xi_1^1=6.5,\xi_2^3=10.5,\xi_3^4=8.5\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$
\begin{aligned}
\widetilde{m}_0^2(A)&=1, & \widetilde{m}_0^3(B)&=1,\\
\widetilde{m}_1^1(A)&=1, & \widetilde{m}_1^2(B)&=1,\\
\widetilde{m}_3^2(A)&=1, & \widetilde{m}_3^3(B)&=1,\\
\widetilde{m}_4^1(A)&=1, & \widetilde{m}_4^2(B)&=1.
\end{aligned}
$$

Consequently, we have the non-zero isotypical defect numbers

$$
\begin{aligned}
&\mathfrak{m}_0^2=-1, && \mathfrak{m}_0^3=1, && \mathfrak{m}_1^1=-1, && \mathfrak{m}_1^2=1,\\
&\mathfrak{m}_3^2=-1, && \mathfrak{m}_3^3=1, && \mathfrak{m}_4^1=-1, && \mathfrak{m}_4^2=1.
\end{aligned}
$$

Hence,

Γ	$\deg^{t}$	# Sols
D_4	$-(D_4^3)-(\mathbb{Z}_4^{t,3})+(D_2^{d,3})-(\widetilde{D}_2^{d,3})+(D_2^3)$ $+(\mathbb{Z}_2^{-,3})-(D_1^{z,3})+(\widetilde{D}_1^{z,3})-2(D_1^3)+2(\widetilde{D}_1^3)$ $-(D_4^{d,2})+(D_2^2)+(\widetilde{D}_1^{z,2})-(D_1^2)$	8
D_5	$-(D_5^3)-(\mathbb{Z}_5^{t_1,3})-(D_1^{z,3})-(D_1^3)+(\mathbb{Z}_1^3)$ $-(\mathbb{Z}_5^{t_2,2})-(D_1^{z,2})-(D_1^2)+(\mathbb{Z}_1^2)$	10
D_6	$-(D_6^{d,3})-(\mathbb{Z}_6^{t_1,3})+(D_3^3)+3(D_2^{d,3})+(D_2^{\hat{d},3})$ $+(\mathbb{Z}_3^{t,3})-2(\widetilde{D}_1^{z,3})-(\widetilde{D}_1^3)-(D_1^3)-2(\mathbb{Z}_2^{-,3})$ $+2(\mathbb{Z}_1^3)-(D_6^{d,2})-(\mathbb{Z}_6^{t_2,2})+(D_3^2)+(D_2^{z,2})$ $+2(D_2^{d,2})+(D_2^2)+(\mathbb{Z}_3^{t,2})-2(\widetilde{D}_1^{z,2})-(\widetilde{D}_1^2)$ $-(D_1^2)-(\mathbb{Z}_2^{-,2})-(\mathbb{Z}_2^2)+2(\mathbb{Z}_1^2)$	11
S_4	$-(S_4^3)+(A_4^3)-(D_4^{d,3})+3(D_3^3)+(D_2^{d,3})$ $+2(D_2^3)+(\mathbb{Z}_4^{c,3})+(\mathbb{Z}_4^{-,3})+(\mathbb{Z}_4^3)+(V_4^{-,3})$ $+(\mathbb{Z}_3^{t,3})-(\mathbb{Z}_3^3)-3(D_1^3)-(\mathbb{Z}_2^{-,3})-2(\mathbb{Z}_2^3)$ $+(\mathbb{Z}_1^3)-(S_4^{-,2})+(A_4^2)-(D_4^{z,2})+3(D_3^{z,2})$ $+(D_2^{d,2})+2(D_2^{z,2})+(\mathbb{Z}_4^{c,2})+(\mathbb{Z}_4^{-,2})+(\mathbb{Z}_4^2)$ $+(V_4^{-,2})+(\mathbb{Z}_3^{t,2})-(\mathbb{Z}_3^2)-3(D_1^{z,2})-(\mathbb{Z}_2^{-,2})$ $-2(\mathbb{Z}_2^2)+(\mathbb{Z}_1^2)$	32
A_5	$-(A_5^3)-(A_4^{t_1,3})-(A_4^{t_2,3})-(A_4^3)+(D_5^{z,3})$ $+3(D_5^3)+2(D_3^{z,3})+4(D_3^3)+3(\mathbb{Z}_5^{t_1,3})+2(\mathbb{Z}_5^{t_2,3})$ $-3(V_4^{-,3})+6(\mathbb{Z}_3^{t,3})+(\mathbb{Z}_3^3)+3(\mathbb{Z}_2^{-,3})-5(\mathbb{Z}_1^3)$ $-(A_4^2)+(D_5^{z,2})+2(D_3^{z,2})+(D_3^2)+(\mathbb{Z}_5^{t_1,2})$ $+2(\mathbb{Z}_5^{t_2,2})-2(V_4^{-,2})+2(\mathbb{Z}_3^{t,2})+(\mathbb{Z}_3^2)+(\mathbb{Z}_2^{-,2})$ $+2(\mathbb{Z}_2^2)-3(\mathbb{Z}_1^2)$	66

Table A4.7. Existence results for the system (10.1) with symmetry group Γ.

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=1}^{\infty}\left(\mathrm{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&=(-1)\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{1,1}}+1\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}\right)\\
&+(-1)\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}+\deg_{\mathcal{V}_{3,3}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{4,1}}+1\cdot\left(\deg_{\mathcal{V}_{4,1}}+\deg_{\mathcal{V}_{4,2}}\right)\\
&=\deg_{\mathcal{V}_{0,3}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{3,3}}+\deg_{\mathcal{V}_{4,2}}.
\end{aligned}
$$

Finally,

$$\begin{aligned} \deg^{t} &= \Theta_3\,[\texttt{showdegree[S4]}\,(1,1,0,1,1,1,0,0,1,0)] \\ &\quad + \Theta_2\,[\texttt{showdegree[S4]}\,(1,1,0,1,1,0,1,0,0,1)]\,. \end{aligned}$$

The dominating orbit types in W are (S_4), (S_4^-), (D_4^d), (D_2^d), $(\mathbb{Z}_4^c) := (\mathbb{Z}_4^{t_1})$, $(\mathbb{Z}_3^t) := (\mathbb{Z}_3^{t_1})$ and (D_4^z). The value of $\deg^{t}$ is listed in Table A4.7.

A4.5.5 Existence in A_5-Symmetric Auto. Newtonian System

Finally, we consider the system (10.1) with the group of symmetrics $G = A_5 \times S^1$, where A_5 denotes the icosahedral group. The A_5-representation $V = \mathbb{R}^{20}$ has the following isotypical decomposition

$$V = \mathcal{V}_0 \oplus (\mathcal{V}_1 \oplus \mathcal{V}_1) \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4,$$

to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, 0, 1, 1, 1)$, and $\sigma(A) = \{\xi_0^0 = 7.5, \xi_1^1 = 4.5, \xi_2^1 = 2.5, \xi_3^2 = 5.5, \xi_4^3 = 4.5 + \sqrt{5}, \xi_5^4 = 4.5 - \sqrt{5}\}$, $\sigma(B) = \{\{\xi_0^0 = 12.5, \xi_1^1 = 9.5, \xi_2^1 = 7.5, \xi_3^2 = 10.5, \xi_4^3 = 9.5 + \sqrt{5}, \xi_5^4 = 9.5 - \sqrt{5}\}\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$\begin{aligned} \widetilde{m}_0^2(A) &= 1, & \widetilde{m}_0^3(B) &= 1, \\ \widetilde{m}_1^1(A) &= 1, & \widetilde{m}_1^3(B) &= 1, \\ \widetilde{m}_1^2(A) &= 1, & \widetilde{m}_1^2(B) &= 1, \\ \widetilde{m}_2^2(A) &= 1, & \widetilde{m}_2^3(B) &= 1, \\ \widetilde{m}_3^2(A) &= 1, & \widetilde{m}_3^3(B) &= 1, \\ \widetilde{m}_4^1(A) &= 1, & \widetilde{m}_4^2(B) &= 1. \end{aligned}$$

Consequently, we have the non-zero isotypical defect numbers

$$\begin{gathered} \mathbf{m}_0^2 = -1, \quad \mathbf{m}_0^3 = 1, \quad \mathbf{m}_1^1 = -1, \quad \mathbf{m}_1^3 = 1, \quad \mathbf{m}_2^2 = -1, \\ \mathbf{m}_2^3 = 1, \quad \mathbf{m}_3^2 = -1, \quad \mathbf{m}_3^3 = 1, \quad \mathbf{m}_4^1 = -1, \quad \mathbf{m}_4^2 = 1. \end{gathered}$$

Hence,

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=1}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= (-1)\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{1,1}}+1\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}\right)\\
&+(-1)\cdot\left(\deg_{\mathcal{V}_{2,1}}+\deg_{\mathcal{V}_{2,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{2,1}}+\deg_{\mathcal{V}_{2,2}}+\deg_{\mathcal{V}_{2,3}}\right)\\
&+(-1)\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}\right)+1\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}+\deg_{\mathcal{V}_{3,3}}\right)\\
&+(-1)\cdot\deg_{\mathcal{V}_{4,1}}+1\cdot\left(\deg_{\mathcal{V}_{4,1}}+\deg_{\mathcal{V}_{4,2}}\right)\\
&=\deg_{\mathcal{V}_{0,3}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}+\deg_{\mathcal{V}_{2,3}}+\deg_{\mathcal{V}_{3,3}}+\deg_{\mathcal{V}_{4,2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\deg^{t} &= \Theta_3\left[\texttt{showdegree[A5]}\,(1,0,1,1,1,1,1,1,1,0)\right]\\
&\quad+\Theta_2\left[\texttt{showdegree[A5]}\,(1,0,1,1,1,0,1,0,0,1)\right].
\end{aligned}
$$

The dominating orbit types: (A_5), (D_3^z), (V_4^-), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$, $(A_4^{t_1})$, $(A_4^{t_2})$ and (D_5^z). The value of $\deg^{t}$ is listed in Table A4.7.

A4.6 Results for Section 10.2

We present the computational examples for $\Gamma = D_n$ and $V = \mathbb{R}^n$ for $n = 6, 8, 10, 12$. Consider the potential $\varphi : V \to \mathbb{R}$ satisfying (A1)–(A3) and (A5). The degeneracy assumptions are listed in Table A4.8.

Γ	$\deg_0$	$\deg_\infty$
D_6	(D_A)+(Y1)	(D_B)+(N1')
D_8	(D_A)+(Y1)	(D'_B)+(N2')
D_{10}	(D'_A)+(N2)	(D_B)+(Y1')
D_{12}	(D'_A)+(N2)	(D'_B)+(Y2')

Table A4.8. Summary of the assumptions in the computational examples.

A4.6.1 Existence in D_6-Symmetric Auto. Degen. Newtonian Sys.

Let $\Gamma = D_6$ and $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4$. Consider the potential $\varphi : V \to \mathbb{R}$ satisfying (A1)–(A3) and (A5) with the matrices A and B being of the type

$$C = \begin{bmatrix} c\,d\,0\,0\,0\,d \\ d\,c\,d\,0\,0\,0 \\ 0\,d\,c\,d\,0\,0 \\ 0\,0\,d\,c\,d\,0 \\ 0\,0\,0\,d\,c\,d \\ d\,0\,0\,0\,d\,c \end{bmatrix}.$$

It can be easily obtained that $\sigma(C) = \{\mu_0 = c+2d, \mu_1 = c+d, \mu_2 = c-d, \mu_4 = c-2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = 8.8$, $d = 4.4$ for A and $c = d = 1.1$ for B, and list eigenvalues of A and B in Table A4.9. Notice that the assumptions ($H3_0$) and ($H4_0$) are satisfied in this case. The dominating orbit types in W are (D_6), (D_6^d), $(\mathbb{Z}_6^{t_1})$, $(\mathbb{Z}_6^{t_2})$, $(D_2^{\hat{d}})$ and (D_2^z).

	c	d	μ_0	μ_1	μ_2	μ_4
A	8.8	4.4	17.6	13.2	4.4	0
B	1.1	1.1	3.3	2.2	0	-1.1

Table A4.9. Eigenvalues of A and B, $\Gamma = D_6$

Using the Table A4.9, we compute the numbers

$$\widetilde{m}_0^4(A) = 1, \quad \widetilde{m}_1^3(A) = 1, \quad \widetilde{m}_2^2(A) = 1,$$
$$\widetilde{m}_0^1(B) = 1, \quad \widetilde{m}_1^1(B) = 1.$$

The value of $\deg^t_{\mathcal{A}}$ is

$$\begin{aligned} &\deg^t_{\mathcal{A}} \\ &= \prod_{\mu \in \sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} * \sum_{j=0}^{s} \sum_{k=0}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \Big(1 \cdot \sum_{l=1}^{4} \deg_{\mathcal{V}_{0,l}} + 1 \cdot \sum_{l=1}^{3} \deg_{\mathcal{V}_{1,l}} + 1 \cdot \sum_{l=1}^{2} \deg_{\mathcal{V}_{2,l}}\Big)\\
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}}\big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{1,2}}\\
&+ \deg_{\mathcal{V}_{2,2}}\big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}}\big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \deg_{\mathcal{V}_{0,4}}\\
&= \Theta_1\,[\texttt{showdegree[D6]}\,(1,1,1,0,0,0,1,1,1,0,0,0)]\\
&+ \Theta_2\,[\texttt{showdegree[D6]}\,(1,1,1,0,0,0,1,1,1,0,0,0)]\\
&+ \Theta_3\,[\texttt{showdegree[D6]}\,(1,1,1,0,0,0,1,1,0,0,0,0)]\\
&+ \Theta_4\,[\texttt{showdegree[D6]}\,(1,1,1,0,0,0,1,0,0,0,0,0)]\\
&= -(\mathbf{D}_6^{\mathbf{d}}) - (\mathbb{Z}_6^{\mathbf{t}_1}) - (\mathbb{Z}_6^{\mathbf{t}_2}) + (\mathbf{D}_2^{\mathbf{z}}) + 3(D_2^{d}) + (\mathbf{D}_2^{\hat{\mathbf{d}}}) + (D_2) - 3(\widetilde{D}_1^{z}) - 2(\widetilde{D}_1)\\
&- 2(D_1^{z}) - 3(D_1) - 2(\mathbb{Z}_2^{-}) - (\mathbb{Z}_2) + 5(\mathbb{Z}_1) - (\mathbf{D}_6^{\mathbf{d},\mathbf{2}}) - (\mathbb{Z}_6^{\mathbf{t}_1,\mathbf{2}}) - (\mathbb{Z}_6^{\mathbf{t}_2,\mathbf{2}})\\
&+ (\mathbf{D}_2^{\mathbf{z},\mathbf{2}}) + 3(D_2^{d,2}) + (\mathbf{D}_2^{\hat{\mathbf{d}},\mathbf{2}}) + (D_2^{2}) - 3(\widetilde{D}_1^{z,2}) - 2(\widetilde{D}_1^{2}) - 2(D_1^{z,2}) - 3(D_1^{2})\\
&- 2(\mathbb{Z}_2^{-,2}) - (\mathbb{Z}_2^{2}) + 5(\mathbb{Z}_1^{2}) - (\mathbf{D}_6^{\mathbf{d},\mathbf{3}}) - (\mathbb{Z}_6^{\mathbf{t}_1,\mathbf{3}}) + 3(D_2^{d,3}) + (\mathbf{D}_2^{\hat{\mathbf{d}},\mathbf{3}}) - 2(\widetilde{D}_1^{z,3})\\
&- (\widetilde{D}_1^{3}) - (D_1^{z,3}) - 2(D_1^{3}) - 2(\mathbb{Z}_2^{-,3}) + 3(\mathbb{Z}_1^{3}) - (\mathbf{D}_6^{\mathbf{d},\mathbf{4}}) + 2(D_2^{d,4})\\
&- (\widetilde{D}_1^{z,4}) - (D_1^{4}) - (\mathbb{Z}_2^{-,4}) + (\mathbb{Z}_1^{4}).
\end{aligned}
$$

Since $Z_0 = \operatorname{Ker} \Lambda \simeq \mathcal{V}_4$, we have the set of all orbit types is $\mathcal{J}(\mathcal{V}_4) = \{(D_6 \times S^1), (D_3 \times S^1)\}$. By (Y1) and Proposition 10.2.7(i), there exist the following nontrivial $(H^{\varphi,l})$-terms in $\deg_0$ (as shown using the above bold symbols):

$$
\begin{gathered}
(D_6^{d}), (\mathbb{Z}_6^{t_1}), (\mathbb{Z}_6^{t_2}), (D_2^{z}), (D_2^{\hat{d}}), (D_6^{d,2}), (\mathbb{Z}_6^{t_1,2}), (\mathbb{Z}_6^{t_2,2}),\\
(D_2^{z,2}), (D_2^{\hat{d},2}), (D_6^{d,3}), (\mathbb{Z}_6^{t_1,3}), (D_2^{\hat{d},3}), (D_6^{d,4}).
\end{gathered}
\tag{A4.5}
$$

On the other hand,

$$\begin{aligned}
&\deg^{t}_{B}\\
&= \prod_{\mu\in\sigma_+(B)}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)} * \sum_{j=0}^{s}\sum_{k=0}^{\infty}\widetilde{m}_j^k(B)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\\
&= \prod_{i=0,1}\deg_{\mathcal{V}_i} * \left(1\cdot\deg_{\mathcal{V}_{0,1}} + 1\cdot\deg_{\mathcal{V}_{1,1}}\right)\\
&= \Theta_1\,[\texttt{showdegree[D6]}\,(1,1,0,0,0,0,1,1,0,0,0,0)]\\
&= -(\mathbf{D}_6^{\mathbf{d}}) - (\mathbb{Z}_6^{\mathbf{t}_1}) - (\mathbf{D}_2^{\mathbf{d}}) - (D_2^{\hat{d}}) + 2(\widetilde{D}_1^{z}) + (\widetilde{D}_1)\\
&+ (D_1^{z}) + 2(D_1) + (\mathbb{Z}_2^{-}) - 3(\mathbb{Z}_1).
\end{aligned}$$

By (N1') and a similar statement as Proposition 10.2.7(ii) for φ satisfying (H0), (H1) and (H4$_0$), we have that $\deg_{\infty}$ does not contain any nontrivial terms as listed in (A4.5) except possibly for (D_6^d), $(\mathbb{Z}_6^{t_1})$ and (D_2^d). Therefore, the following orbit types will appear in the value $\deg_{\infty} - \deg_{0}$:

$$(D_6^{d,2}), (D_6^{d,3}), (D_6^{d,4}), (\mathbb{Z}_6^{t_1,2}), (\mathbb{Z}_6^{t_1,3}), (\mathbb{Z}_6^{t_2}), (\mathbb{Z}_6^{t_2,2}), (D_2^{z}), (D_2^{z,2}), (D_2^{\hat{d},2}), (D_2^{\hat{d},3}).$$

Conclusion: Under the assumptions (A1)—(A3), (D$_A$), (D$_B$) and (A5), by Theorem 10.1.3$_d$, there exist at least 11 nonstationary solutions of (10.1). To be more specific, there are

- 1 nonstationary solution with least symmetry $(D_6^{d,4})$;
- 2 nonstationary solutions with least symmetries $(\mathbb{Z}_6^{t_1,3})$;
- 2 nonstationary solutions with least symmetries $(\mathbb{Z}_6^{t_2,2})$;
- 3 nonstationary solutions with least symmetries $(D_2^{z,2})$ and
- 3 nonstationary solutions with least symmetries $(D_2^{\hat{d},3})$.

A4.6.2 Existence in D_8-Symmetric Auto. Degen. Newtonian Sys.

Let $\Gamma = D_8$ and $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5$. Consider the potential $\varphi : V \to \mathbb{R}$ satisfying (A1)—(A3) and (A5) with the matrices A and B being of the type

$$C = \begin{bmatrix}
c & d & 0 & 0 & 0 & 0 & 0 & d\\
d & c & d & 0 & 0 & 0 & 0 & 0\\
0 & d & c & d & 0 & 0 & 0 & 0\\
0 & 0 & d & c & d & 0 & 0 & 0\\
0 & 0 & 0 & d & c & d & 0 & 0\\
0 & 0 & 0 & 0 & d & c & d & 0\\
0 & 0 & 0 & 0 & 0 & d & c & d\\
d & 0 & 0 & 0 & 0 & 0 & d & c
\end{bmatrix}.$$

It can be easily obtained that $\sigma(C) = \{\mu_0 = c + 2d, \mu_1 = c + \sqrt{2}d, \mu_2 = c, \mu_3 = c - \sqrt{2}d, \mu_5 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = 4\sqrt{2}$, $d = 4$ for A and $c = 3$, $d = \sqrt{2}$ for B, and list eigenvalues of A and B in Table A4.10* . Notice that the assumptions (H3$_0$) and (H4$_l$) (for $l_\infty = 1$) are satisfied in this case. The dominating orbit types in W are (D_8), (D_8^d), $(\mathbb{Z}_8^{t_1})$, $(\mathbb{Z}_8^{t_2})$, $(\mathbb{Z}_8^{t_3})$, $(\widetilde{D}_4^d)$.

	c	d	μ_0	μ_1	μ_2	μ_3	μ_5
A	$4\sqrt{2}$	4	13.7	11.3	5.7	0	-2.3
B	3	$\sqrt{2}$	5.8	5	3	1	0.2

Table A4.10. Eigenvalues of A and B, $\Gamma = D_8$

Using the Table A4.10, we compute the numbers

$$\widetilde{m}_0^3(A) = 1, \quad \widetilde{m}_1^3(A) = 1, \quad \widetilde{m}_2^2(A) = 1,$$
$$\widetilde{m}_0^2(B) = 1, \quad \widetilde{m}_1^2(B) = 1, \quad \widetilde{m}_2^1(B) = 1, \widetilde{m}_3(l_\infty^2) = 1.$$

Compute the value of $\deg_{\mathcal{A}}^t$ by

$$\begin{aligned}
&\deg_{\mathcal{A}}^t \\
&= \prod_{\mu\in\sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} * \sum_{j=0}^{s}\sum_{k=0}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \\
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \Big(1 \cdot \sum_{l=1}^{3} \deg_{\mathcal{V}_{0,l}} + 1 \cdot \sum_{l=1}^{3} \deg_{\mathcal{V}_{1,l}} + 1 \cdot \sum_{l=1}^{2} \deg_{\mathcal{V}_{2,l}}\Big) \\
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}}\big) \\
&+ \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{2,2}}\big) \\
&+ \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}}\big)
\end{aligned}$$

---

* The eigenvalues are evaluated only up to 10^{-1}, which is sufficient for determining the numbers $\widetilde{m}_j^k$ for the computations of degree.

$$
\begin{aligned}
&= \Theta_1\,[\texttt{showdegree[D8]}\,(1,1,1,0,0,0,0,1,1,1,0,0,0,0)]\\
&+ \Theta_2\,[\texttt{showdegree[D8]}\,(1,1,1,0,0,0,0,1,1,1,0,0,0,0)]\\
&+ \Theta_3\,[\texttt{showdegree[D8]}\,(1,1,1,0,0,0,0,1,1,0,0,0,0,0)]\\
&= -(\mathbf{D_8}) - (\boldsymbol{\widetilde{D}_4^{d}}) - (D_4^d) - (\boldsymbol{\mathbb{Z}_8^{t_1}}) - (\boldsymbol{\mathbb{Z}_8^{t_2}}) + (\widetilde{D}_2^z) + 2(\widetilde{D}_2^d) + 2(\widetilde{D}_2)\\
&+ (D_2^z) + 2(D_2^d) + 2(D_2) + (\mathbb{Z}_4^d) - 2(\widetilde{D}_1^z) - 3(\widetilde{D}_1) - 2(D_1^z) - 3(D_1)\\
&- (\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 5(\mathbb{Z}_1) - (\mathbf{D_8^2}) - (\boldsymbol{\widetilde{D}_4^{d,2}}) - (D_4^{d,2}) - (\boldsymbol{\mathbb{Z}_8^{t_1,2}}) - (\boldsymbol{\mathbb{Z}_8^{t_2,2}})\\
&+ (\widetilde{D}_2^{z,2}) + 2(\widetilde{D}_2^{d,2}) + 2(\widetilde{D}_2^2) + (D_2^{z,2}) + 2(D_2^{d,2}) + 2(D_2^2) + (\mathbb{Z}_4^{d,2})\\
&- 2(\widetilde{D}_1^{z,2}) - 3(\widetilde{D}_1^2) - 2(D_1^{z,2}) - 3(D_1^2) - (\mathbb{Z}_2^{-,2}) - 3(\mathbb{Z}_2^2) + 5(\mathbb{Z}_1^2)\\
&- (\mathbf{D_8^3}) - (\boldsymbol{\mathbb{Z}_8^{t_1,3}}) + (\widetilde{D}_2^{d,3}) + (\widetilde{D}_2^3) + (D_2^{d,3}) + (D_2^3) - (\widetilde{D}_1^{z,3})\\
&- 2(\widetilde{D}_1^3) - (D_1^{z,3}) - 2(D_1^3) - (\mathbb{Z}_2^{-,3}) - (\mathbb{Z}_2^3) + 3(\mathbb{Z}_1^3).
\end{aligned}
$$

Since $Z_0 = \operatorname{Ker} A \simeq \mathcal{V}_3$, we have the set of all orbit types is $\mathcal{J}(\mathcal{V}_3) = \{(D_8 \times S^1), (D_1 \times S^1), (\widetilde{D}_1 \times S^1), (\mathbb{Z}_1 \times S^1)\}$. By (Y1) and Proposition 10.2.7(i), there exist the following nontrivial $(H^{\varphi,l})$-terms in $\deg_0$ (as shown using the above bold symbols):

$$
(D_8), (\widetilde{D}_4^d), (\mathbb{Z}_8^{t_1}), (\mathbb{Z}_8^{t_2}), (D_8^2), (\widetilde{D}_4^{d,2}), (\mathbb{Z}_8^{t_1,2}), (\mathbb{Z}_8^{t_2,2})(D_8^3), (\mathbb{Z}_8^{t_1,3}). \qquad \text{(A4.6)}
$$

On the other hand,

$$
\begin{aligned}
&\deg_B^{\,l}\\
&= \deg_B^{\,0} * \Big(\sum_{j=0}^{s}\sum_{k=1}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s} \widetilde{m}_j(l_\infty^2) \sum_{l=1}^{l_\infty - 1} \deg_{\mathcal{V}_{j,l}}\Big)\\
&\overset{(l_\infty = 1)}{=} \prod_{i=0,1,2,3,5} \deg_{\mathcal{V}_i} * \big(1 \cdot (\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{0,2}}) + 1 \cdot (\deg_{\mathcal{V}_{1,1}}\\
&+ \deg_{\mathcal{V}_{1,2}}) + 1 \cdot \deg_{\mathcal{V}_{2,1}}\big)\\
&= \Theta_1\,[\texttt{showdegree[D8]}\,(1,1,1,1,0,1,0,1,1,1,0,0,0,0)]\\
&+ \Theta_2\,[\texttt{showdegree[D8]}\,(1,1,1,1,0,1,0,1,1,0,0,0,0,0)]\\
&= -(\mathbf{D_8}) - (\boldsymbol{\widetilde{D}_4^{d}}) + (D_4^d) - (\boldsymbol{\mathbb{Z}_8^{t_1}}) - (\boldsymbol{\mathbb{Z}_8^{t_2}}) + (\widetilde{D}_2^z)\\
&+ (\widetilde{D}_2^d) + 2(\widetilde{D}_2) - (D_2^z) - (D_2^d) - 2(D_2) + (\mathbb{Z}_4^{t_1})\\
&+ (\mathbb{Z}_4^d) - (\mathbf{D_8^2}) + (D_4^2) - (\boldsymbol{\mathbb{Z}_8^{t_1,2}}) + (\widetilde{D}_2^{d,2}) - (\widetilde{D}_2^2)\\
&- (D_2^{d,2}) - (D_2^2) + (\mathbb{Z}_4^{t_1,2}).
\end{aligned}
$$

By (N2') and a similar statement as Proposition 10.2.8(ii) for φ satisfying (H0)—(H2) and (H4$_l$), we have that $\deg_\infty$ does not contain any nontrivial terms as listed in (A4.7) except possibly for (D_8), $(\widetilde{D}_4^d)$, $(\mathbb{Z}_8^{t_1})$, $(\mathbb{Z}_8^{t_2})$, (D_8^2) and $(\mathbb{Z}_8^{t_1,2})$. Moreover, since $Z_\infty \simeq \mathcal{V}_{3,1}$, we have that $\mathcal{J}(Z_\infty) = \{(D_8 \times S^1), (\mathbb{Z}_8^{t_3}), (D_2^d), (\widetilde{D}_2^d), (\mathbb{Z}_2^d)\}$. Therefore, the following orbit types $(H^{\varphi,l})$ will appear in the value $\deg_\infty - \deg_0$:

$$(D_8^3), (\widetilde{D}_4^{d,2}), (\mathbb{Z}_8^{t_1,3}), (\mathbb{Z}_8^{t_2,2}). \tag{A4.7}$$

Conclusion: Under the assumptions (A1)—(A3), (D$_A$), (D$_B$) and (A5), by Theorem 10.1.3$_d$, there exist at least 7 nonstationary solutions of (10.1). To be more specific, there are

- 1 nonstationary solution with least symmetry (D_8^3);
- 2 nonstationary solutions with least symmetries $(\widetilde{D}_4^{d,2})$;
- 2 nonstationary solutions with least symmetries $(\mathbb{Z}_8^{t_2,2})$ and
- 2 nonstationary solutions with least symmetries $(\mathbb{Z}_8^{t_1,3})$.

A4.6.3 Existence in D_{10}-Symmetric Auto. Degen. Newtonian Sys.

Let $\Gamma = D_{10}$ and $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6$. Consider the potential $\varphi : V \to \mathbb{R}$ satisfying (A1)—(A3) and (A5) with the matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}.$$

It can be easily obtained that $\sigma(C) = \{\mu_0 = c + 2d, \mu_1 = c + 2d\cos\frac{\pi}{5}, \mu_2 = c + 2d\cos\frac{2\pi}{5}, \mu_3 = c + 2d\cos\frac{3\pi}{5}, \mu_4 = c + 2d\cos\frac{4\pi}{5}, \mu_6 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = -2$, $d = 3$ for A and $c = 4$, $d = 2(\cos(2\pi/5))^{-1}$ for B, and list eigenvalues of A and B in Table A4.11. Notice that the assumptions (H3$_l$) and (H4$_0$) are satisfied in this case (for $l_0 = 2$). The dominating orbit types in W are (D_{10}), (D_{10}^d), $(\mathbb{Z}_{10}^{t_1})$, $(\mathbb{Z}_{10}^{t_2})$, $(\mathbb{Z}_{10}^{t_3})$, $(\mathbb{Z}_{10}^{t_4})$, $(D_2^{\hat{d}})$, (D_2^z).

	c	d	μ_0	μ_1	μ_2	μ_3	μ_4	μ_6
A	-2	3	4	2.9	-0.1	-3.9	-6.9	-8
B	4	$2(\cos(2\pi/5))^{-1}$	17	14.5	8	0	-6.5	-8.9

Table A4.11. Eigenvalues of A and B, $\Gamma = D_{10}$

Using the Table A4.11, we compute the numbers

$$\widetilde{m}_0(l_0^2) = 1, \quad \widetilde{m}_1^1(A) = 1, \quad \widetilde{m}_0^4(B) = 1, \quad \widetilde{m}_1^3(B) = 1, \quad \widetilde{m}_2^2(B) = 1.$$

Compute the value of $\deg^{l}_{B}$

$$\deg^{l}_{B}$$

$$= \prod_{\mu\in\sigma_+(B)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} * \sum_{j=0}^{s}\sum_{k=0}^{\infty} \widetilde{m}_j^k(B) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}$$

$$= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \Big(1 \cdot \sum_{l=1}^{4} \deg_{\mathcal{V}_{0,l}} + 1 \cdot \sum_{l=1}^{3} \deg_{\mathcal{V}_{1,l}} + 1 \cdot \sum_{l=1}^{2} \deg_{\mathcal{V}_{2,l}}\Big)$$

$$= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}}\big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{1,2}}$$

$$+ \deg_{\mathcal{V}_{2,2}}\big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}}\big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} * \deg_{\mathcal{V}_{0,4}}$$

$$= \Theta_1\,[\texttt{showdegree[D10]}\,(1,1,1,0,0,0,0,0,1,1,1,0,0,0,0,0)]$$

$$+ \Theta_2\,[\texttt{showdegree[D10]}\,(1,1,1,0,0,0,0,0,1,1,1,0,0,0,0,0)]$$

$$+ \Theta_3\,[\texttt{showdegree[D10]}\,(1,1,1,0,0,0,0,0,1,1,0,0,0,0,0,0)]$$

$$+ \Theta_4\,[\texttt{showdegree[D10]}\,(1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0)]$$

$$= -(\mathbf{D_{10}}) - (\mathbb{Z}_{\mathbf{10}}^{\mathbf{t_1}}) - (\mathbb{Z}_{\mathbf{10}}^{\mathbf{t_2}}) + (D_2^d) + (\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}}}) + (\mathbf{D}_{\mathbf{2}}^{\mathbf{z}}) + 3(D_2) - 2(\widetilde{D}_1^z)$$

$$- 3(\widetilde{D}_1) - 2(D_1^z) - 3(D_1) - (\mathbb{Z}_2^-) - 2(\mathbb{Z}_2) + 5(\mathbb{Z}_1) - (\mathbf{D}_{\mathbf{10}}^{\mathbf{2}}) - (\mathbb{Z}_{\mathbf{10}}^{\mathbf{t_1,2}})$$

$$- (\mathbb{Z}_{\mathbf{10}}^{\mathbf{t_2,2}}) + (D_2^{d,2}) + (\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}},\mathbf{2}}) + (\mathbf{D}_{\mathbf{2}}^{\mathbf{z},\mathbf{2}}) + 3(D_2^2) - 2(\widetilde{D}_1^{z,2}) - 3(\widetilde{D}_1^2) - 2(D_1^{z,2})$$

$$- 3(D_1^2) - (\mathbb{Z}_2^{-,2}) - 2(\mathbb{Z}_2^2) + 5(\mathbb{Z}_1^2) - (\mathbf{D}_{\mathbf{10}}^{\mathbf{3}}) - (\mathbb{Z}_{\mathbf{10}}^{\mathbf{t_1,3}}) + (D_2^{d,3}) + (\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}},\mathbf{3}})$$

$$+ 2(D_2^3) - (\widetilde{D}_1^{z,3}) - 2(\widetilde{D}_1^3) - (D_1^{z,3}) - 2(D_1^3) - (\mathbb{Z}_2^{-,3}) - (\mathbb{Z}_2^3) + 3(\mathbb{Z}_1^3)$$

$$- (\mathbf{D}_{\mathbf{10}}^{\mathbf{4}}) + 2(D_2^4) - (\widetilde{D}_1^4) - (D_1^4) - (\mathbb{Z}_2^4) + (\mathbb{Z}_1^4).$$

Since $Z_\infty = \operatorname{Ker} B \simeq \mathcal{V}_3$, we have the set of all orbit types is $\mathcal{J}(\mathcal{V}_3) = \{(D_{10}\times S^1), (D_1 \times S^1), (\widetilde{D}_1 \times S^1), (\mathbb{Z}_1 \times S^1)\}$. By (Y1') and a similar statement as Proposition 10.2.7(i), there exist the following nontrivial $(H^{\varphi,l})$-terms in $\deg_\infty$ (as shown using the above bold symbols):

$$\begin{aligned}&(D_{10}), (\mathbb{Z}_{10}^{t_1}), (\mathbb{Z}_{10}^{t_2}), (D_2^{\hat{d}}), (D_2^{z}), (D_{10}^2), (\mathbb{Z}_{10}^{t_1,2}), (\mathbb{Z}_{10}^{t_2,2}),\\ &(D_2^{\hat{d},2}), (D_2^{z,2}), (D_{10}^3), (\mathbb{Z}_{10}^{t_1,3}), (D_2^{\hat{d},3}), (D_{10}^4).\end{aligned} \tag{A4.8}$$

On the other hand,

$$\begin{aligned}&\deg_{\mathcal{A}}^{l}\\ &= \deg_{\mathcal{A}}^{0} * \Big(\sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^k(A)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s}\widetilde{m}_j(l_0^2)\sum_{l=1}^{l_0-1}\deg_{\mathcal{V}_{j,l}}\Big)\\ &\overset{(l_0=2)}{=} \prod_{i=0}^{1}\deg_{\mathcal{V}_i} * \big(1\cdot\deg_{\mathcal{V}_{1,1}} + 1\cdot\deg_{\mathcal{V}_{0,1}}\big)\\ &= \Theta_1\,[\texttt{showdegree[D10]}\,(1,1,0,0,0,0,0,0,1,1,0,0,0,0,0,0)]\\ &= -(\mathbf{D_{10}}) - (\mathbb{Z}_{\mathbf{10}}^{\mathbf{t_1}}) - (D_2^{d}) - (\mathbf{D}_{\mathbf{2}}^{\hat{\mathbf{d}}}) + (\widetilde{D}_1^{z})\\ &+ 2(\widetilde{D}_1) + (D_1^{z}) + 2(D_1) + (\mathbb{Z}_2^{-}) - 3(\mathbb{Z}_1).\end{aligned}$$

Since $Z_0 \simeq \mathcal{V}_{0,2}$, we have that $\mathcal{J}(Z_0) = \{(D_{10}\times S^1), (D_{10}^2)\}$. By (N2), except for possibly (D_{10}), $(\mathbb{Z}_{10}^{t_1})$, $(D_2^{\hat{d}})$ and (D_{10}^2), every orbit types listed in (A4.8) will appear in the value of $\deg_\infty - \deg_0$, namely:

$$\begin{aligned}&(\mathbb{Z}_{10}^{t_2}), (D_2^{z}), (\mathbb{Z}_{10}^{t_1,2}), (\mathbb{Z}_{10}^{t_2,2}), (D_2^{\hat{d},2}),\\ &(D_2^{z,2}), (D_{10}^3), (\mathbb{Z}_{10}^{t_1,3}), (D_2^{\hat{d},3}), (D_{10}^4).\end{aligned}$$

Conclusion: Under the assumptions (A1)—(A3), (D_A), (D_B) and (A5), by Theorem 10.1.3_d, there exist altogether at least 15 nonstationary solutions of (10.1). To be more specific, there are

- 2 nonstationary solutions with least symmetries $(\mathbb{Z}_{10}^{t_2,2})$;
- 5 nonstationary solutions with least symmetries $(D_2^{z,2})$;
- 2 nonstationary solutions with least symmetries $(\mathbb{Z}_{10}^{t_1,3})$;
- 5 nonstationary solutions with least symmetries $(D_2^{\hat{d},3})$ and
- 1 nonstationary solution with least symmetry (D_{10}^4).

A4.6.4 Existence in D_{12}-Symmetric Auto. Degen. Newtonian Sys.

Let $\Gamma = D_{12}$ and $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4 \oplus \mathcal{V}_5 \oplus \mathcal{V}_7$. Consider the potential $\varphi : V \to \mathbb{R}$ satisfying (A1)—(A3) and (A5) with the matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}.$$

It can be easily obtained that $\sigma(C) = \{\mu_0 = c + 2d, \mu_1 = c + \sqrt{3}d, \mu_2 = c + d, \mu_3 = c, \mu_4 = c - d, \mu_5 = c - \sqrt{3}d, \mu_7 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = -2$, $d = 2\sqrt{3}$ for A and $c = 3$, $d = \sqrt{3}$ for B, and list eigenvalues of A and B in Table A4.12. Notice that the assumptions (H3$_l$) (for $l_0 = 2$) and (H4$_l$) (for $l_\infty = 3$) are satisfied in this case. The dominating orbit types in W are (D_{12}), (D_{12}^d), $(\mathbb{Z}_{12}^{t_1})$, $(\mathbb{Z}_{12}^{t_2})$, $(\mathbb{Z}_{12}^{t_3})$, $(\mathbb{Z}_{12}^{t_4})$, $(\mathbb{Z}_{12}^{t_5})$, $(\tilde{D}_6^d)$, (D_4^z), $(D_4^{\hat{d}})$.

	c	d	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_7
A	-1	2.5	4	3.3	1.5	-1	-3.5	-5.3	-6
B	9	3.7	16.4	15.4	12.7	9	5.3	2.6	1.6

Table A4.12. Eigenvalues of A and B, $\Gamma = D_{12}$

Using the Table A4.12, we compute the numbers

$$\widetilde{m}_0(l_0^2) = 1, \quad \widetilde{m}_1^1(A) = 1, \quad \widetilde{m}_1^2(A) = 1, \quad \widetilde{m}_0^4(B) = 1, \quad \widetilde{m}_1^3(B) = 1,$$
$$\widetilde{m}_2^3(B) = 1, \widetilde{m}_3(l_\infty^2) = 1, \quad \widetilde{m}_4^2(B) = 1, \quad \widetilde{m}_5^1(B) = 1, \quad \widetilde{m}_7^1(B) = 1.$$

Compute the value of $\deg_B^l$

$$
\begin{aligned}
&\deg^{\,l}_{\mathcal{B}} \\
&= \deg^{\,0}_{\mathcal{A}} * \Big(\sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}^{k}_{j}(\Lambda)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s}\widetilde{m}_{j}(l^{2}_{\infty})\sum_{l=1}^{l_\infty-1}\deg_{\mathcal{V}_{j,l}}\Big) \\
&\overset{l_\infty=3}{=} \prod_{i\in\{0,\dots,5,7\}} \deg_{\mathcal{V}_i} * \Big(1\cdot\sum_{l=1}^{4}\deg_{\mathcal{V}_{0,l}} + 1\cdot\sum_{l=1}^{3}\deg_{\mathcal{V}_{1,l}} + 1\cdot\sum_{l=1}^{3}\deg_{\mathcal{V}_{2,l}} \\
&+ 1\cdot\sum_{l=1}^{2}\deg_{\mathcal{V}_{3,l}} + 1\cdot\sum_{l=1}^{2}\deg_{\mathcal{V}_{4,l}} + 1\cdot\deg_{\mathcal{V}_{5,1}} + 1\cdot\deg_{\mathcal{V}_{7,1}}\Big) \\
&= \prod_{i\in\{0,\dots,5,7\}} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}} + \deg_{\mathcal{V}_{3,1}} + \deg_{\mathcal{V}_{4,1}} + \deg_{\mathcal{V}_{5,1}} \\
&+ \deg_{\mathcal{V}_{7,1}}\big) + \prod_{i\in\{0,\dots,5,7\}} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{2,2}} + \deg_{\mathcal{V}_{3,2}} + \deg_{\mathcal{V}_{4,2}}\big) \\
&+ \prod_{i\in\{0,\dots,5,7\}} \deg_{\mathcal{V}_i} * \big(\deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}} + \deg_{\mathcal{V}_{2,3}}\big) + \prod_{i\in\{0,\dots,5,7\}} \deg_{\mathcal{V}_i} * \deg_{\mathcal{V}_{0,4}} \\
&= \Theta_1\,[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,1,1,1,1,1,0,1,0)] \\
&+ \Theta_2\,[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,1,1,1,1,0,0,0,0)] \\
&+ \Theta_3\,[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,1,1,0,0,0,0,0,0)] \\
&+ \Theta_4\,[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,0,0,0,0,0,0,0,0)] \\
&= -(\mathbf{D_{12}}) - (\mathbf{D^{d}_{12}}) - (\mathbb{Z}^{\mathbf{t_1}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_2}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_3}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_4}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_5}}_{\mathbf{12}}) - (\widetilde{\mathbf{D}}^{\mathbf{d}}_{\mathbf{6}}) \\
&+ (D^{d}_{6}) + 2(D_6) + (\mathbb{Z}^{d}_{6}) + 2(\mathbb{Z}^{t_1}_{6}) + 2(\mathbb{Z}^{t_2}_{6}) + 3(D_4) + 3(D^{d}_{4}) + (\mathbf{D}^{\mathbf{z}}_{\mathbf{4}}) \\
&+ (\mathbf{D}^{\hat{\mathbf{d}}}_{\mathbf{4}}) + 2(\widetilde{D}_3) - 3(D_3) - (D^{z}_{3}) + 2(\widetilde{D}^{z}_{3}) - 2(\mathbb{Z}^{d}_{4}) - 2(\mathbb{Z}_4) - 2(\widetilde{D}^{d}_{2}) \\
&- 2(\widetilde{D}_2) + 2(D^{d}_{2}) - 2(D_2) - 3(\widetilde{D}^{z}_{2}) + 4(\widetilde{D}_1) - 4(D_1) + 4(\widetilde{D}^{z}_{1}) - 4(D^{z}_{1}) \\
&+ 4(\mathbb{Z}_2) - (\mathbf{D}^{\mathbf{2}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_1,2}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_2,2}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_3,2}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_4,2}}_{\mathbf{12}}) - (\widetilde{\mathbf{D}}^{\mathbf{d,2}}_{\mathbf{6}}) + (D^{d,2}_{6}) \\
&+ (D^{2}_{6}) + (D^{d,2}_{4}) + 3(D^{2}_{4}) + (\mathbf{D}^{\hat{\mathbf{d}},\mathbf{2}}_{\mathbf{4}}) + (\mathbf{D}^{\mathbf{z,2}}_{\mathbf{4}}) + (\mathbb{Z}^{d,2}_{6}) + (\mathbb{Z}^{t_1,2}_{6}) + 2(\mathbb{Z}^{t_2,2}_{6}) \\
&+ 2(\widetilde{D}^{2}_{3}) - 2(D^{2}_{3}) + (\widetilde{D}^{z,2}_{3}) - (D^{z,2}_{3}) - (\mathbb{Z}^{d,2}_{4}) - 2(\mathbb{Z}^{2}_{4}) - (\widetilde{D}^{d,2}_{2}) - 3(\widetilde{D}^{2}_{2}) \\
&+ (D^{d,2}_{2}) - (D^{2}_{2}) - 2(\widetilde{D}^{z,2}_{2}) + 3(\widetilde{D}^{2}_{1}) - 3(D^{2}_{1}) + 3(\widetilde{D}^{z,2}_{1}) - 3(D^{z,2}_{1}) + 3(\mathbb{Z}^{2}_{2}) \\
&- (\mathbf{D}^{\mathbf{3}}_{\mathbf{12}}) + (D^{3}_{6}) - (\mathbb{Z}^{\mathbf{t_1,3}}_{\mathbf{12}}) - (\mathbb{Z}^{\mathbf{t_2,3}}_{\mathbf{12}}) + (D^{d,3}_{4}) + (\mathbf{D}^{\hat{\mathbf{d}},\mathbf{3}}_{\mathbf{4}}) + 2(D^{3}_{4}) + (\widetilde{D}^{3}_{3}) \\
&- (D^{3}_{3}) + (\mathbb{Z}^{t_1,3}_{6}) + (\mathbb{Z}^{t_2,3}_{6}) - (\widetilde{D}^{z,3}_{2}) - (\widetilde{D}^{d,3}_{2}) + (D^{d,3}_{2}) - 2(\widetilde{D}^{3}_{2}) - (D^{3}_{2}) \\
&- (\mathbb{Z}^{d,3}_{4}) - (\mathbb{Z}^{3}_{4}) + 2(\widetilde{D}^{z,3}_{1}) - 2(D^{z,3}_{1}) + 2(\widetilde{D}^{3}_{1}) - 2(D^{3}_{1}) + 2(\mathbb{Z}^{3}_{2}) - (\mathbf{D}^{\mathbf{4}}_{\mathbf{12}}) \\
&+ (D^{4}_{6}) + 2(D^{4}_{4}) + (\widetilde{D}^{4}_{3}) - (D^{4}_{3}) - (\widetilde{D}^{4}_{2}) - (D^{4}_{2}) - (\mathbb{Z}^{2}_{4}) + (\mathbb{Z}^{4}_{2}).
\end{aligned}
$$

Since $Z_\infty = \mathcal{V}_{3,3} \simeq \mathcal{U}_3$, we have the set of all orbit types is $\mathcal{J}(\mathcal{U}_3) = \{(\mathbb{Z}_{12}^{t_3}), (D_6^d), (\widetilde{D}_6^d), (\mathbb{Z}_6^d)\}$. By (Y2') and Proposition 10.2.8(i), there exist the following nontrivial terms in $\deg_\infty$ (as shown using the above bold symbols):

$$\begin{aligned}&(D_{12}), (D_{12}^d), (\mathbb{Z}_{12}^{t_1}), (\mathbb{Z}_{12}^{t_2}), (\mathbb{Z}_{12}^{t_3}), (\mathbb{Z}_{12}^{t_4}), (\mathbb{Z}_{12}^{t_5}), (\widetilde{D}_6^d), (D_4^z),\\ &(D_4^{\hat{d}}), (D_{12}^2), (\mathbb{Z}_{12}^{t_1,2}), (\mathbb{Z}_{12}^{t_2,2}), (\mathbb{Z}_{12}^{t_3,2}), (\mathbb{Z}_{12}^{t_4,2}), (\widetilde{D}_6^{d,2}),\\ &(D_4^{\hat{d},2}), (D_4^{z,2}), (D_{12}^3), (\mathbb{Z}_{12}^{t_1,3}), (\mathbb{Z}_{12}^{t_2,3}), (D_4^{\hat{d},3}), (D_{12}^4).\end{aligned} \tag{A4.9}$$

On the other hand,

$$\begin{aligned}&\deg_{\mathcal{A}}^{t}\\ &= \deg_{\mathcal{A}}^{0} * \Big(\sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^k(A)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s}\widetilde{m}_j(l_0^2)\sum_{l=1}^{l_0-1}\deg_{\mathcal{V}_{j,l}}\Big)\\ &\overset{l_0=2}{=} \prod_{i=0}^{2}\deg_{\mathcal{V}_i} * \big(1\cdot\deg_{\mathcal{V}_{0,1}} + 1\cdot\deg_{\mathcal{V}_{1,1}} + 1\cdot\deg_{\mathcal{V}_{2,1}}\big)\\ &= \Theta_1\,[\texttt{showdegree[D12]}\,(1,1,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0)]\\ &= -(\mathbf{D_{12}}) - (\mathbb{Z}_{\mathbf{12}}^{\mathbf{t_1}}) - (\mathbb{Z}_{\mathbf{12}}^{\mathbf{t_2}}) - (D_4^d) - (\mathbf{D}_{\mathbf{4}}^{\hat{\mathbf{d}}}) + (\widetilde{D}_2^z) + (D_2^z) + (\widetilde{D}_2^d)\\ &+ (D_2^d) + 2(\widetilde{D}_2) + 2(D_2) + (\mathbb{Z}_4^d) - 2(\widetilde{D}_1^z) - 2(D_1^z) - 3(\widetilde{D}_1) - 3(D_1)\\ &- (\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 5(\mathbb{Z}_1).\end{aligned}$$

Since $Z_0 = \mathcal{V}_{0,2}$, we have the set of all orbit types is $\mathcal{J}(Z_0) = \{(D_{12} \times S^1), (D_{12})\}$. By (N2) and Proposition 10.2.8(ii), except for possibly (D_{12}), $(\mathbb{Z}_{12}^{t_1})$, $(\mathbb{Z}_{12}^{t_2})$ and $(D_4^{\hat{d}})$, every orbit types listed in (A4.9) will appear in $\deg_\infty - \deg_0$, namely

$$\begin{gathered}(D_{12}^d), (\mathbb{Z}_{12}^{t_3}), (\mathbb{Z}_{12}^{t_4}), (\mathbb{Z}_{12}^{t_5}), (\widetilde{D}_6^d), (D_4^z), (D_{12}^2), (\mathbb{Z}_{12}^{t_1,2}),\\ (\mathbb{Z}_{12}^{t_2,2}), (\mathbb{Z}_{12}^{t_3,2}), (\mathbb{Z}_{12}^{t_4,2}), (\widetilde{D}_6^{d,2}), (D_4^{\hat{d},2}), (D_4^{z,2}),\\ (D_{12}^3), (\mathbb{Z}_{12}^{t_1,3}), (\mathbb{Z}_{12}^{t_2,3}), (D_4^{\hat{d},3}), (D_{12}^4).\end{gathered}$$

Conclusion: Under the assumptions (A1)—(A3), (D_A), (D_B) and (A5), by Theorem $10.1.3_d$, there exist altogether at least 20 nonstationary solutions of (10.1). To be more specific, there are

- 1 nonstationary solution with least symmetry (D_{12}^4);

- 1 nonstationary solution with least symmetry (D_{12}^{d});
- 2 nonstationary solutions with least symmetries ($\mathbb{Z}_{12}^{t_1,3}$);
- 2 nonstationary solutions with least symmetries ($\mathbb{Z}_{12}^{t_2,3}$);
- 2 nonstationary solutions with least symmetries ($\mathbb{Z}_{12}^{t_3,2}$);
- 2 nonstationary solutions with least symmetries ($\mathbb{Z}_{12}^{t_4,2}$);
- 2 nonstationary solutions with least symmetries ($\mathbb{Z}_{12}^{t_5}$);
- 2 nonstationary solutions with least symmetries ($\widetilde{D}_{6}^{d,2}$);
- 3 nonstationary solutions with least symmetries ($D_{4}^{\hat{d},3}$) and
- 3 nonstationary solutions with least symmetries ($D_{4}^{z,2}$).

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